THE TERWILLIGER ALGEBRAS OF THE GROUP ASSOCIATION SCHEMES OF S_5 AND A_5

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Abstract. Terwilliger proposed a method for studying commutative association schemes by introducing a non-commutative, semi-simple $\mathbb{C}$-algebra, whose structure reflects the combinatorial nature of the corresponding scheme, and applied the method to the P and Q polynomial schemes. In this paper, we continue the initial investigation of Bannai and Munemasa of the Terwilliger algebras of group association schemes. In particular, we determine the structure of the Terwilliger algebras for the group schemes of S_5 and A_5.

§ 1. Introduction

Terwilliger algebras were introduced in [7] as a method for studying commutative association schemes. The structure of these algebras reflect the combinatorial nature of the corresponding schemes. Terwilliger's method proved very effective in studying schemes with many vanishing intersection numbers and Krein parameters, in particular, the P and Q polynomial schemes and their relatives.

In this paper, we determine the structure of the Terwilliger algebras of the group association schemes of S_5 and A_5. We first recall some important definitions and refer the reader to [6], [7], and [8] for details on general Terwilliger algebras and to [1] for association schemes.

Definition 1. Let G be a finite group. Let $C_0, C_1, \dots, C_d$ be the conjugacy classes of G. Define the relations $R_i (i = 0, 1, \dots, d)$ on G by

$$(x, y) \in R_i \Leftrightarrow yx^{-1} \in C_i.$$

Then $\mathfrak{X}(G) = (G, \{R_i\}_{0 \le i \le d})$ is a commutative association scheme of class d called the *group association scheme of* G.

Let $A_0, A_1, \dots, A_d$ denote the adjacency matrices of the relations of the scheme. Then

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$$

and these matrices form a basis for the Bose-Mesner algebra $\mathfrak{A}$, a semi-simple subalgebra of $Mat_G(\mathbb{C})$. The intersection numbers p_{ij}^k of the group association

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scheme $\mathfrak{X}(G)$ are given by $p_{ij}^k = |\{(x,y) \in C_i \times C_j \,|\, xy = z, \text{ for a fixed } z \in C_k\}|$. The algebra $\mathfrak{A}$ has a second basis $E_0, E_1, \ldots, E_d$ of primitive idempotents, and

$$E_i \circ E_j = \frac{1}{|G|} \sum_{k=0}^{d} q_{ij}^k E_k,$$

where $\circ$ denotes Hadamard (entry-wise) multiplication. The non-negative real numbers q_{ij}^k are called the Krein parameters.

For each $i = 0, 1, \ldots, d$, let E_i^* and A_i^* be the diagonal matrices of size $|G| \times |G|$ defined respectively by

$$(E_i^*)_{x,x} = \begin{cases} 1 & \text{if } x \in C_i \\ 0 & \text{if } x \notin C_i \end{cases} \qquad (x \in G),$$

$$(A_i^*)_{x,x} = |G|(E_i)_{e,x} \quad (x \in G,\ e = \text{identity of } G).$$

Then $\mathfrak{A}^* = \langle E_0^*, \ldots, E_d^* \rangle = \langle A_0^*, \ldots, A_d^* \rangle$ is a commutative, semi-simple subalgebra of $Mat_G(\mathbb{C})$ called the dual Bose-Mesner subalgebra of $\mathfrak{X}(G)$.

Definition 2. The *Terwilliger algebra* $T(G)$ of the group scheme $\mathfrak{X}(G)$ is the subalgebra of $Mat_G(\mathbb{C})$ generated by $\mathfrak{A}$ and $\mathfrak{A}^*$.

The algebra $T(G) = \langle \mathfrak{A}, \mathfrak{A}^* \rangle = \langle A_i, E_i^* \mid 0 \le i \le d \rangle$ is non-commutative and semi-simple (it is closed under the conjugate-transpose map). It is of natural interest to determine the combinatorial and group-theoretical properties of $T(G)$ such as its dimension, and its irreducible complex representations, which Bannai and Munemasa have initiated in [2].

In this paper, we determine the dimensions of $T(S_5)$ and $T(A_5)$, obtain explicitly complete sets of primitive central idempotents, and as our main result, determine the following Wedderburn decompositions:

$$T(S_5) \cong \mathcal{M}_1 \oplus \mathcal{M}_1 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_5 \oplus \mathcal{M}_5 \oplus \mathcal{M}_6 \oplus \mathcal{M}_7,$$

and

$$T(A_5) \cong \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_5 \oplus \mathcal{M}_5,$$

where $\mathcal{M}_i$ denotes the full matrix algebra over $\mathbb{C}$ of degree i.

§2. Preliminaries

In this section we collect key results in [2] and [6]. Background information on groups and their representations can be found in standard references like [3] and [5]. For the rest of the paper we assume the notation of the previous section unless otherwise stated. We will write T to indicate general Terwilliger algebras.

The intersection numbers provide some information about the structure of the Terwilliger algebra. The following relations in T were found by Terwilliger ([6, Lemma 3.2]):

$$E_i^* A_j E_k^* = 0 \quad \text{iff} \quad p_{ij}^k = 0 \quad (0 \le i, j, k \le d)$$
$$E_i A_j^* E_k = 0 \quad \text{iff} \quad q_{ij}^k = 0 \quad (0 \le i, j, k \le d).$$

For $T(G)$, $E_i^* A_j E_k^*$ is the (0,1)-matrix of size $|G| \times |G|$ with rows and columns indexed by the elements of G arranged by conjugacy classes, and whose (C_i, C_k) block (rows indexed by elements of C_i and columns by C_k) is identical to the (C_i, C_k) block of A_j.

We now review some results of Bannai and Munemasa in [2]. Define T_o and T_o^* to be the linear span of the $E_i^* A_j E_k^*$ and $E_i A_j^* E_k$, $(0 \le i, j, k \le d)$, respectively. Then the dimensions of T_o and T_o^* equal the number of nonvanishing p_{ij}^k and q_{ij}^k, respectively. Furthermore,

$$T = \langle T_o \rangle = \langle T_o^* \rangle.$$

For the intersection numbers of the group scheme $\mathfrak{X}(G)$, the following is well known:

$$p_{ij}^k = \frac{|C_i||C_j|}{|G|} \sum_{\chi \in Irr(G)} \frac{\chi(u_i)\chi(u_j)\overline{\chi(u_k)}}{\chi(1)}.$$

An analogous formula holds for q_{ij}^k (see ([1]).

Now consider the action of G on itself via conjugation. Let $\mathcal{O}$ be the totality of G-orbits in $G \times G$ (under component-wise action). Then denote by $\widetilde{T} = \widetilde{T}(G)$ the centralizer algebra of the permutation representation of G affording $(G, \mathcal{O})$. (In the language of D. Higman [4], $(G, \mathcal{O})$ is a coherent configuration of degree $|G|$ and rank $|\mathcal{O}|$.). The adjacency matrices of the graphs $(G, \mathcal{O}_i)$, where $\mathcal{O}_i \in \mathcal{O}$, is a basis of $\widetilde{T}(G)$. Then since $A_i \in \widetilde{T}$ and $E_i^* \in \widetilde{T}$, for all i, we have

$$T(G) \subseteq \widetilde{T}.$$

The dimension of $\widetilde{T}$ which is the rank of the configuration $(G, \mathcal{O})$ can then be computed using group theory. Hence we have the following bounds for the dimension of $T(G)$:

Proposition 1. *Let T(G) be the Terwilliger algebra of the group association scheme of G. Then:*

$$|\{(i,j,k) \,|\, p_{ij}^k \neq 0,\ 0 \le i,j,k \le d\}| \le dim\, T(G) \le \sum_{i=0}^{d} |C_G(u_i)|,$$

where the u_i, $(i = 0, 1, \ldots, d)$ is a set of representatives for the conjugacy classes of G and $C_G(u_i)$ is the centralizer in G of u_i.

Note. We also have $|\{(i,j,k) \,|\, q_{ij}^k \neq 0, 0 \le i,j,k \le d\}| \le dim\, T(G)$.

Remarks. 1. In general, these dimensions do not coincide. But there are many groups for which $dim\, T_o = dim\, \widetilde{T}$ (see [2]). For example, all abelian groups and all dihedral groups. Also, if G_1 and G_2 are groups satisfying the above condition, then so does $G_1 \times G_2$.

2. We also mention another interesting result in [2]. First, we state a definition. Let $\mathfrak{X} = (X, \{R_i\})$ be a commutative association scheme, and let $R_i(x) = \{y \in X \,|\, (x,y) \in R_i\}$. The scheme $\mathfrak{X}$ is called triply regular if the size of the set $R_i(x) \cap R_j(y) \cap R_k(z)$ depends only on the set $\{i, j, k, l, m, n\}$ where $(x,y) \in R_l$, $(x,z) \in R_m$,

and $(y, z) \in R_n$. Then Munemasa proves that $\mathfrak{X}$ is triply regular if and only if $T = T_o$.

3. The observations above yield the following information: (i) for S_3: $dim\,T_o = dim\,\widetilde{T} = dim\,T = 11$; (ii) for A_4: $dim\,T_o = dim\,T_o^* = dim\,T = 19$; and (iii) for S_4: $dim\,T_o^* = dim\,\widetilde{T} = dim\,T = 43$. In this paper, we study the groups S_5 and A_5, which provide the first "non-trivial" cases for the family of symmetric and alternating groups.

§3. The algebra $\widetilde{T}$

In this section we examine the algebra $\widetilde{T}$ in more detail. Since $\widetilde{T}$ is a semisimple algebra, we have $\widetilde{T} = \oplus\, \widetilde{T}\tilde{\varepsilon}_i$, where the $\tilde{\varepsilon}_i$ are primitive central idempotents ($\tilde{\varepsilon}_i{}^2 = \tilde{\varepsilon}_i \neq 0$), given by (see [5, Chapter II, § 1]):

$$\tilde{\varepsilon}_i = \frac{|\chi_i(1)|}{|G|} \sum_{g \in G} \overline{\chi_i(g)} g^*, \quad (\chi_i \in Irr(G)).$$

Here, $g \mapsto g^*$ is the permutation representation given by $(g^*)_{x,y} = (\delta_{g^{-1}xg,y})_{x,y}$. Each minimal two-sided ideal $\widetilde{T}\tilde{\varepsilon}_i$ is isomorphic to a full matrix algebra.

The degree d_i of the irreducible complex representation afforded by $\tilde{\varepsilon}_i$ equals the multiplicity of the character χ_i as an irreducible constituent of the permutation character π (under the action of conjugation). Let $\pi = \sum d_i \chi_i, \chi_i \in Irr(G)$. Then

$$d_i = <\pi, \chi_i> = \sum_{j=0}^{d} \overline{\chi_i(u_j)}, \quad (u_j \in C_j),$$

where $<\,,\,>$ is the usual inner product on the space of class functions on G and u_j is a representative of the class C_j. From this we see that the number of inequivalent irreducible representations of $\widetilde{T}(G)$ equals the number of non-zero row sums in the character table of G. In particular, from the information above, we obtain for the groups S_5 and A_5 the following:

Proposition 2. *Let $\widetilde{T}(S_5)$ and $\widetilde{T}(A_5)$ be as previously defined. Then:*

(1) $\widetilde{T}(S_5) \cong \mathcal{M}_1 \oplus \mathcal{M}_3 \oplus \mathcal{M}_4 \oplus \mathcal{M}_5 \oplus \mathcal{M}_5 \oplus \mathcal{M}_6 \oplus \mathcal{M}_7$, *and*

(2) $\widetilde{T}(A_5) \cong \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_5 \oplus \mathcal{M}_5$,

where $\mathcal{M}_i$ denotes the full matrix algebra over $\mathbb{C}$ of degree i.

§4. The Terwilliger Algebras of $\mathfrak{X}(S_5)$ and $\mathfrak{X}(A_5)$

We present in this section the main results of the paper. We start with the following result.

Theorem 1. *The dimensions of T_o^*, T_o, T, and $\widetilde{T}$ for the groups S_5 and A_5 are given by the following:*

		T_o^*	T_o	T	$\widetilde{T}$
(1)	S_5 :	143	124	155	161
(2)	A_5 :	65	71	73	77

Proof. The dimensions of T_o^*, T_o, and $\widetilde{T}$ are easily computed using the results in §2. Next, by direct calculation, we obtain for each of A_5 and S_5 a linearly independent set closed under multiplication which spans the set of all products $E_i^* A_j E_k^* \cdot E_k^* A_l E_m^* = E_i^* A_j E_k^* A_l E_m^*$ of the size asserted above. Since $T = \langle E_i^* A_j E_k^* \rangle$, the set is a basis of T. $\square$

Remark. The basis of T obtained for each group consists of (0,1)-matrices. Hence both $T(S_5)$ and $T(A_5)$ are closed under Hadamard multiplication. Each generator $E_i^* A_j E_k^*$ of T_o is either contained in the basis obtained for T or gives rise to "new" basis elements of T. The basis for $\widetilde{T}$ arises from that of T in a similar manner.

For the rest of the paper, we fix the ordering of the conjugacy classes of the two groups as follows:

(1) S_5 conjugacy class representatives and orders:

	C_0	C_1	C_2	C_3	C_4	C_5	C_6
u_i :	(1)	(12)	(123)	(12)(34)	(1234)	(12)(345)	(12345)
$\lvert C_i \rvert$:	1	10	20	15	30	20	24

(2) A_5 conjugacy class representatives and orders:

	C_0	C_1	C_2	C_3	C_4
u_i :	(1)	(123)	(12)(34)	(12345)	$(12345)^2$
$\lvert C_i \rvert$:	1	20	15	12	12

We provide the following two matrices, each indexed by the conjugacy classes in the order assumed earlier, whose entries indicate the distribution of the basis elements of T_o, T, and $\widetilde{T}$ respectively. For example, the entry $3+2+1$ in the (C_4, C_6)-position for the group S_5 indicates that two additional basis elements of T arise from the set $\{E_4^* A_j E_6^*\}_{0 \le j \le 6}$, and further, an extra basis element of $\widetilde{T}$ comes from those of T in the same location. The leftmost number in the (C_i, C_k)-entry indicates the number of nonzero $E_i^* A_j E_k^*$ of T_o. Both matrices are symmetric, so we omit the entries below the diagonal.

$$S_5 : \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 3 & 3 & 3 & 3 & 3 & 2 \\ & & 4+1 & 3 & 3+2 & 3+2 & 3+1 \\ & & & 4 & 3+1+1 & 3 & 3 \\ & & & & 4+3+2 & 3+2 & 3+2+1 \\ & & & & & 4+1 & 3+1 \\ & & & & & & 4+4 \end{pmatrix}$$

$$A_5 \; : \; \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & 5+1 & 4+0+1 & 3 & 3 \\ & & 5+1+2 & 4 & 4 \\ & & & 4 & 4 \\ & & & & 4 \end{pmatrix}$$

We briefly describe the idea behind the rest of our investigation, which uses mainly the fact that T is a semi-simple algebra.

First, let $\mathcal{Z}(T)$ denote the center of T. Since T contains the diagonal matrices E_i^*, we observe that $\mathcal{Z}(T)$ consists of block diagonal matrices, and that

$$\mathcal{Z}(T) \subseteq \bigoplus_{i=0}^{d} \mathcal{Z}(E_i^* T E_i^*).$$

Recall that the dimension of $\mathcal{Z}(T)$ equals the number of minimal two-sided ideals of T. We let $s = dim\, \mathcal{Z}(T)$.

We then exhibit a set $\{\varepsilon_i \,|\, 1 \leq i \leq s\}$ of primitive central idempotents for T (i.e., ${\varepsilon_i}^2 = \varepsilon_i \neq 0$, $\varepsilon_i \varepsilon_j = \delta_{ij}\varepsilon_i$, $1_T = \sum_{i=1}^{s} \varepsilon_i$, and $\varepsilon_i \in Z(T)$), from which we obtain the decomposition of T into a sum of minimal two-sided ideals, each isomorphic to matrix algebras. Thus,

$$T \;=\; \bigoplus_{i=1}^{s} T\varepsilon_i \;\cong\; \bigoplus_{i=1}^{s} \mathcal{M}_{d_i}(\mathbb{C})\,,$$

where $\mathcal{M}_{d_i}(\mathbb{C})$ is a full matrix algebra over $\mathbb{C}$ of degree d_i.

Lemma 1. *The dimensions of the centers of $T(S_5)$ and $T(A_5)$ are as follows:*

(1) $dim\, \mathcal{Z}(T(S_5)) = 8$,

(2) $dim\, \mathcal{Z}(T(A_5)) = 6$.

Proof. The result is obtained by explicitly determining a basis for the center. This is done as follows. First we determine a basis for $\mathcal{Z}(E_i^* T E_i^*)$ for all $i = 0, 1, \ldots, d$. Then let $\{b_j\}$ be the union of the bases of the $\mathcal{Z}(E_i^* T E_i^*)$ for all i. Thus if $y \in \mathcal{Z}(T)$, we can write $y = \sum c_j b_j \,, c_j \in \mathbb{Z}$. The system of linear equations $\{x_i y = y x_i\}$, ranging over all elements x_i in the basis of T (obtained earlier) is then solved, and yields up to scalars, the required basis for $\mathcal{Z}(T)$. $\square$

Let $\{e_i \,|\, 1 \leq i \leq s\}$ be the basis of the center obtained above, so that

$$\mathcal{Z}(T) \;=\; \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_s.$$

Then

$$e_i e_j \;=\; \sum_{k=1}^{s} t_{ij}^k e_k, \quad t_{ij}^k \in \mathbb{Z}.$$

Define the matrices $B_i \;=\; (t_{ij}^k)$, for $i = 1, \ldots, s$ by

$$(B_i)_{j,k} \;:=\; t_{ij}^k.$$

Since these matrices mutually commute, they are simultaneously diagonalised by a non-singular matrix, say P. Thus, for each $i = 1, \dots s$, we have

$$P^{-1}B_iP = \begin{pmatrix} v_1(i) & & \\ & \ddots & \\ & & v_s(i) \end{pmatrix}.$$

Next, define the matrix $M = (v_i(j))$ by

$$(M)_{i,j} := v_i(j).$$

Then we see that the primitive central idempotents $\varepsilon_1, \dots, \varepsilon_s$ are given by the following matrix equation:

$$(\varepsilon_1, \dots, \varepsilon_s) = (e_1, \dots, e_s)M^{-1}.$$

In the next theorem, for brevity we shall write $e_{ijk} := E_i^* A_j E_k^*$. The notation n_{ijk} or n_{ij_lk} indicates a basis element of T obtained from the element e_{ijk}.

Theorem 2. *(i) The primitive central idempotents of $T(S_5)$ and $T(A_5)$ are given by the following :*

(1) $T(S_5)$:

$$\begin{aligned}
\varepsilon_1 &= \tfrac{1}{24}(e_{606} + e_{626} - 2n_{626} - e_{636} - e_{666} - 2n_{66_a6} - 2n_{66_b6}) \\
\varepsilon_2 &= \tfrac{1}{12}(2e_{404} - e_{434} - n_{434} + e_{464} - n_{464}) \\
\varepsilon_3 &= \tfrac{1}{12}(4e_{303} - 2e_{323} - 2e_{333} + e_{363}) \\
&\quad + \tfrac{1}{24}(4e_{404} - 2e_{424} + e_{434} + 3n_{434} + e_{464} - 3n_{464}) \\
&\quad + \tfrac{1}{24}(5e_{606} - e_{626} - e_{636} - e_{666} + 6n_{66_a6} + 6n_{66_b6} + 6n_{66_c6}) \\
\varepsilon_4 &= \tfrac{1}{10}(3e_{202} + e_{222} - 4n_{222} - e_{232}) \\
&\quad + \tfrac{1}{20}(4e_{404} + 2e_{424} - 2n_{424} + e_{434} - 5n_{434} - e_{464} - n_{464}) \\
&\quad + \tfrac{1}{10}(3e_{505} - 3n_{525} + e_{535} - e_{565}) \\
&\quad + \tfrac{1}{2}(e_{606} - n_{66_c6}) \\
\varepsilon_5 &= \tfrac{1}{10}(2e_{202} - e_{222} - n_{222} + e_{232}) \\
&\quad + \tfrac{1}{30}(4e_{404} - 3e_{424} + 3n_{424} + e_{434} - 5n_{434} - e_{464} + 4n_{464}) \\
&\quad + \tfrac{1}{10}(2e_{505} - 2n_{525} - e_{535} + e_{565}) \\
\varepsilon_6 &= \tfrac{1}{15}(6e_{101} + e_{121} - 4e_{131}) \\
&\quad + \tfrac{1}{30}(6e_{202} + e_{222} + 5n_{222} + e_{232} - 4e_{262}) \\
&\quad + \tfrac{1}{15}(4e_{303} - e_{323} + 4e_{333} - e_{363}) \\
&\quad + \tfrac{1}{30}(4e_{404} - e_{424} + 5n_{424} - e_{434} + 5n_{434} - e_{464}) \\
&\quad + \tfrac{1}{30}(6e_{505} - 4e_{525} + 10n_{525} + e_{535} + e_{565})
\end{aligned}$$

$$\begin{aligned}
\varepsilon_7 &= \tfrac{1}{6}(3e_{101} - e_{121} + e_{131}) \\
&\quad + \tfrac{1}{12}(3e_{202} - e_{222} + 4n_{222} - e_{232} + e_{262}) \\
&\quad + \tfrac{1}{12}(4e_{303} + 2e_{323} - 2e_{333} - e_{363}) \\
&\quad + \tfrac{1}{24}(4e_{404} + 2e_{424} - 4n_{424} - e_{434} + 5n_{434} - e_{464} + 3n_{464}) \\
&\quad + \tfrac{1}{12}(3e_{505} + e_{525} + 2n_{525} - e_{535} - e_{565}) \\
&\quad + \tfrac{1}{24}(5e_{606} - e_{626} + 2n_{626} + e_{636} - e_{666} - 4n_{66_a6} - 4n_{66_b6} + 6n_{66_c6})
\end{aligned}$$

$$\begin{aligned}
\varepsilon_8 &= e_{000} + \tfrac{1}{10}(e_{101} + e_{121} + e_{131}) \\
&\quad + \tfrac{1}{20}(e_{202} + e_{222} + e_{232} + e_{262}) \\
&\quad + \tfrac{1}{15}(e_{303} + e_{323} + e_{333} + e_{363}) \\
&\quad + \tfrac{1}{30}(e_{404} + e_{424} + e_{434} + e_{464}) \\
&\quad + \tfrac{1}{20}(e_{505} + e_{525} + e_{535} + e_{565}) \\
&\quad + \tfrac{1}{24}(e_{606} + e_{626} + e_{636} + e_{666})
\end{aligned}$$

(2) $T(A_5)$ *:*

$$\varepsilon_1 = \tfrac{1}{10}(2e_{101} - e_{111} - n_{111} + e_{121})$$

$$\begin{aligned}
\varepsilon_2 &= \tfrac{1}{30}(6e_{101} + e_{111} + 5n_{111} + e_{121} - 4e_{131} - 4e_{141}) \\
&\quad + \tfrac{1}{15}(4e_{202} - e_{212} + 4e_{222} - e_{232} - e_{242})
\end{aligned}$$

$$\begin{aligned}
\varepsilon_3 &= \tfrac{1}{20}(3e_{101} + e_{111} - 4n_{111} - e_{121} + \sqrt{5}e_{131} - \sqrt{5}e_{141}) \\
&\quad + \tfrac{1}{20}(5e_{303} - \sqrt{5}e_{313} + \sqrt{5}e_{333} - 5e_{343}) \\
&\quad + \tfrac{1}{20}(5e_{404} + \sqrt{5}e_{414} - 5e_{434} - \sqrt{5}e_{444})
\end{aligned}$$

$$\begin{aligned}
\varepsilon_4 &= \tfrac{1}{20}(3e_{101} + e_{111} - 4n_{111} - e_{121} - \sqrt{5}e_{131} + \sqrt{5}e_{141}) \\
&\quad + \tfrac{1}{20}(5e_{303} + \sqrt{5}e_{313} - \sqrt{5}e_{333} - 5e_{343}) \\
&\quad + \tfrac{1}{20}(5e_{404} - \sqrt{5}e_{414} - 5e_{434} + \sqrt{5}e_{444})
\end{aligned}$$

$$\begin{aligned}
\varepsilon_5 &= \tfrac{1}{12}(3e_{101} - e_{111} + 4n_{111} - e_{121} + e_{131} + e_{141}) \\
&\quad + \tfrac{1}{3}(2e_{202} - e_{222}) \\
&\quad + \tfrac{1}{12}(5e_{303} - e_{313} - e_{333} + 5e_{343}) \\
&\quad + \tfrac{1}{12}(5e_{404} - e_{414} + 5e_{434} - e_{444})
\end{aligned}$$

$$\begin{aligned}
\varepsilon_6 &= e_{000} + \tfrac{1}{20}(e_{101} + e_{111} + e_{121} + e_{131} + e_{141}) \\
&\quad + \tfrac{1}{15}(e_{202} + e_{212} + e_{222} + e_{232} + e_{242}) \\
&\quad + \tfrac{1}{12}(e_{303} + e_{313} + e_{333} + e_{343}) \\
&\quad + \tfrac{1}{12}(e_{404} + e_{414} + e_{434} + e_{444})
\end{aligned}$$

(ii) The degrees of the irreducible complex representations afforded by the idempo-

tents are given below:

(1) $T(S_5)$:

ε_i	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7	ε_8
deg ε_i	1	1	3	3	5	5	6	7

(2) $T(A_5)$:

ε_i	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6
deg ε_i	1	2	3	3	5	5

Proof. The idempotents were obtained using the method described before the theorem. To determine the degrees d_i afforded by each ε_i, we know that $T\varepsilon_i \cong \mathcal{M}_{d_i}(\mathbb{C})$, hence ${d_i}^2 = dim\, T\varepsilon_i$ equals the number of linearly independent elements in the set $\{x_j\varepsilon_i\}$ where the x_j are the basis elements of T. We then count this number for each idempotent ε_i. $\square$

Theorem 3. *(Structure Theorem for $T(S_5)$ and $T(A_5)$)*

$$\begin{aligned} &(1) \quad T(S_5) \cong \mathcal{M}_1 \oplus \mathcal{M}_1 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_5 \oplus \mathcal{M}_5 \oplus \mathcal{M}_6 \oplus \mathcal{M}_7, \\ &(2) \quad T(A_5) \cong \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_5 \oplus \mathcal{M}_5, \end{aligned}$$

where $\mathcal{M}_i$ is a full matrix algebra over $\mathbb{C}$ of degree i.

Proof. This is immediate from the preceding theorem. $\square$

Remarks. 1. Let $V = \mathbb{C}^{|G|}$ (column vectors) and let $<,>$ denote the form $< u, v >:= u^t\overline{v}$, $(u, v) \in V$, where t denotes transpose and $\overline{v}$ denotes the complex conjugate of v. Then $(V, <,>)$ is the standard module of $\mathcal{M}at_{|G|}(\mathbb{C})$ (acting on V by left multiplication), and V decomposes into an orthogonal direct sum

$$V = \varepsilon_1 V \oplus \cdots \oplus \varepsilon_s V.$$

Here $\varepsilon_s V$ corresponds to the idempotent $\varepsilon_s = \frac{1}{|G|}\sum_{g\in G} g^*$ of highest degree (i.e., ε_8 for $T(S_5)$ and ε_6 for $T(A_5)$) and is an irreducible $T(G)$-submodule called the principle submodule.

2. Recall that the primitive central idempotents of $\widetilde{T}(G)$ are given by $\tilde{\varepsilon}_i = \frac{|\chi_i(1)|}{|G|}\sum_{g\in G} \overline{\chi_i(g)}g^*$, $\chi_i \in Irr(G)$. We observe that in the case of $T(S_5)$,

$$\varepsilon_2 + \varepsilon_3 \,=\, \tilde{\varepsilon}_\chi \,=\, \frac{|\chi(1)|}{120} \sum_{g\in S_5} \chi(g)g^* \,,$$

where χ is the irreducible character of degree 5 of S_5 corresponding to the partition [2,2,1]. The rest of the idempotents of $T(S_5)$ and $\widetilde{T}(S_5)$ are identical, i.e., $\varepsilon_i = \tilde{\varepsilon}_i$, after renumbering.

Similarly, for $T(A_5)$, we have

$$\varepsilon_1 + \varepsilon_2 \,=\, \tilde{\varepsilon}_\psi \,=\, \frac{|\psi(1)|}{60} \sum_{g\in A_5} \psi(g)g^*,$$

where ψ is the irreducible character of A_5 of degree four obtained from the standard permutation representation afforded by the doubly transitive action of A_5 on a set of five points. And as in the case of S_5, after renumbering, $\varepsilon_i = \tilde{\varepsilon}_i$, for the rest of the idempotents.

We would like to explicitly determine later the irreducible representations afforded by these idempotents.

3. The observations in this paper hopefully give insight to the structure of the Terwilliger algebras of group association schemes for the general case, which we intend to investigate next.

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