Type II Codes over $\mathbb{F}_2 + u\mathbb{F}_2$ and Applications to Hermitian Modular Forms

Eiichi Bannai[*], Masaaki Harada[†], Tomoyoshi Ibukiyama[‡]
Akihiro Munemasa[§] and Manabu Oura[¶]

1 Introduction

The theorem of Gleason [14] says that the weight enumerators $W_C(x, y)$ of binary Type II codes C (that is, self-dual and doubly even codes over the binary field $\mathbb{F}_2$) are in the ring of polynomial invariants $\mathbb{C}[x, y]^G$ of the action of the finite group

$$G = \left\langle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \right\rangle \subset GL(2, \mathbb{C}) \tag{1}$$

of order 192, and that this invariant ring is isomorphic to the polynomial ring generated by two homogeneous polynomials of degrees 8 and 24. Note that we can take the weight enumerators of the extended Hamming $[8, 4, 4]$ code and the extended Golay $[24, 12, 8]$ code as the generators of the invariant ring. The theorem of Broué and Enguehard [5] says that elliptic modular forms of weight $n/2$ are obtained from the homogeneous weight enumerators $W_C(x, y)$ of binary Type II codes C, by substituting the Jacobi theta series $\theta_3(2\tau)$ and $\theta_2(2\tau)$ for x and y, respectively. The elliptic modular form obtained in this way from $W_C(x, y)$ coincides with the theta series of the lattice obtained by Construction A (see [8, p. 183]) from the binary Type II code C we started with. Moreover, it is shown by Ozeki [23] and Runge [30], that all the elliptic modular forms are obtained from the invariant ring $\mathbb{C}[x, y]^H$ of the index 2 subgroup

$$H = \left\langle \frac{1+i}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \right\rangle$$

[*]Faculty of Mathematics, Kyushu University, Fukuoka 812–8581, Japan

[†]Department of Mathematical Sciences, Yamagata University, Yamagata 990–8560, Japan

[‡]Department of Mathematics, Graduate School of Sciences, Osaka University, Osaka 560–0043, Japan

[§]Graduate School of Information Sciences, Tohoku University, Sendai 980–8579, Japan. This work was done while he was at Kyushu university

[¶]Division of Mathematics, Liberal Arts and Sciences, School of Medicine, Sapporo Medical University, Sapporo 060–8556, Japan. This work was done while he was at Kyushu university and supported by a grant from the Japan Society for the Promotion of Science.

of the group G above, by the same substitution by the Jacobi theta series $\theta_3(2\tau)$ and $\theta_2(2\tau)$ for x and y, respectively.

This theme of exploring the connections between codes and lattices, between the weight enumerators of codes and the theta series of the lattices, as well as between the polynomial invariants of finite groups and the modular forms has been extensively studied, and generalized to many directions, by many authors. For a recent survey on these subjects, see Rains and Sloane [25], Bannai [2].

For example,

(i) By considering multi-weight enumerators (of binary Type II codes) to get Siegel modular forms (Duke [10], Herrmann [16], Runge [26, 27, 30]),

(ii) By considering certain joint weight enumerators of binary Type II codes (Jacobi polynomials in the sense of Ozeki [23]) to get Jacobi forms (Bannai and Ozeki [4], Runge [28]),

(iii) By considering Lee weight enumerators of self-dual codes over $\mathbb{F}_p$ to get Hilbert modular forms (see Ebeling [11]).

Also, it is now fashionable to consider codes not just over finite fields but over certain finite rings, say $\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}$ etc. Actually, this approach is very fruitful.

The purpose of this paper is to push this study further. Namely, we consider codes over the finite ring $R = \mathbb{F}_2 + u\mathbb{F}_2 = \mathbb{Z}[i]/2\mathbb{Z}[i]$ of 4 elements, where $u^2 = 0$. We define Type II codes over R and their symmetrized biweight enumerators, and establish the relationship with symmetric Hermitian modular forms. It seems that this finite ring R fits the study of Hermitian modular forms very well, and we were aware of this fact for several years. In the meantime, the paper which explicitly states the connection between Type II codes and even unimodular Hermitian lattices was published Dougherty, Harada, Gaborit and Solé [9], together with the classification of Type II codes of lengths 4 and 8 over R. (Note that the length of a Type II code over R is always a multiple of 4.) In this paper, we connect the algebra of symmetrized biweight enumerators of Type II codes over the ring R with the algebra of polynomial invariants by a certain finite group action, and then directly construct symmetrized Hermitian modular forms from the polynomial invariants by the finite group, by substituting certain theta series for the indeterminates of the polynomial. This will be explained in Section 8 of the present paper, after the preliminary sections Section 2 and Section 3. One of the main purposes of this paper is to give the classification of Type II codes over R for the lengths 12 and 16. In this classification, we utilize the following key method which is explained in Section 4. That is, there is a one to one correspondence between the equivalence clases of Type II codes of length n over R with a prescribed binary image C, and the conjugacy classes of fixed-foint-free involutions in the automorphism group of the binary Type II code C of length $2n$. The actual classifications of Type II codes over R of lengths 12 and 16 are obtained in Section 5, by applying the method of Section 4 to the known classification of binary Type II codes of lengths 24 and 32, which were obtained by Pless and Sloane [24] and Conway, Pless and Sloane [7], respectively. There are 82, 1894 Type II codes over R up to equivalence for the lengths 12, 16, respectively.

In the rest of this paper, we consider symmetric Hermitian modular forms of degree 2. The structure of the algebra of symmetric Hermitian forms of degree 2 was studied by Freitag [12], Nagaoka [21] and Runge [29]. They gave the generators of this ring very explicitly. Namely, they gave the 6 generators of weights 4, 8, 10, 12, 12, and 16. (Their generators are slightly different, but are related to each other.) A relation among the generators was explicitly given in Nagaoka [21]. We note that the existence of such a relation was stated in [12] and [29]. Runge [29] also gave some general description of the ring of modular forms depending on a choice of modular embeddings determined by "Picard type". As we show in Section 8, the symmetrized biweight enumerator of a Type II code of length n over R gives a symmetric Hermitian modular form of degree 2 and of weight n. We can calculate the symmetric Hermitain modular forms of degree 2 corresponding to each of the Type II codes in the classification. The results are tabulated in Table 4 for lengths 8 and 12. Then we can identify the generators of weight 4, 8, 12, 12 and 16 of Freitag (and Nagaoka) in terms of the symmetrized biweight enumerators of the codes over R. The remaining generator, a symmetric Hermitain modular form of weight 10, can not be directly obtained from a code, because there is no Type II code of length 10. However, we will show that it is in fact obtained as an invariant of a finite group $G \subset GL(10, \mathbb{C})$ of order 737280. This group can be thought of a subgroup of index 2 in the natural matrix group which leaves the symmetrized biweight enumerators invariant. This is very similar to the situation where the Eisenstein series E_6 is obtained as an invariant of the subgroup H of index 2 of the group G in (1). We will describe the invariant of degree 10 of $G \subset GL(10, \mathbb{C})$ explicitly at the end of Section 8. It is implicit in [12, 21] that the dimension of the space A_k^s of symmetric Hermitian modular forms of degree 2 and of weight k is given as follows:

$$\sum_{k=0}^{\infty} \dim A_k^s t^k = \frac{1+t^{16}}{(1-t^4)(1-t^8)(1-t^{10})(1-t^{12})^2}.$$

It is expected that the 6 generators (due to Freitag and Nagaoka) will in fact reflect the Cohen–Macaulay structure of the algebra of symmetric Hermitian modular forms of degree 2 suggested by the above mentioned dimension formula. The fact that Freitag–Nagaoka generators actually reflect the Cohen-Macaulay structure was mentioned as a conjecture in Nagaoka's paper [21, p. 547], although it was known to Freitag before. Since the proof of this fact (i.e. there exists the unique fundamental relation among generators), as well as the correctness of the above dimension formula, was not explicitly stated in the literature, we give two proofs of this fact in Section 7, for the convenience of the reader. Here we treated only those modular forms with trivial character. The full Hermitian modular group has the character v of order two. The ring of Hermitian modular forms of weight $2k$ with character v^k was determined by Hermann [15]. If $k \equiv 0 \pmod 4$, then $v^{k/2} = 1$, and we can express the generators of modular forms without character by generators of modular forms with characters at least in principle. Since we actually need some calculation to obtain these expressions, we shall explain it in this paper. We see also that, if we use these expressions, we can get the relation given by Nagaoka at once. The authors also thank Professor Nagaoka for his help in understanding more on Hermitian modular forms. A part of the results obtained in this paper was announced in several places by some of the authors. For example, see [3, 22].

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2 Preliminaries

This section collects the required notations and definitions. Some basic results which are used in this paper are also given.

2.1 Type II Codes

In this paper, we study Type II codes over a commutative ring $\mathbb{F}_2+u\mathbb{F}_2 = \{0, 1, u, 1+u\}$ with $u^2 = 0$ of order 4. $\mathbb{F}_2+u\mathbb{F}_2$ is isomorphic to the quotient ring $\mathbb{Z}[i]/2\mathbb{Z}[i] = \{0, 1, i, u = 1+i\}$. We shall consider the ring using whichever form is more convenient. Throughout this paper, we denote by R the rings $\mathbb{F}_2 + u\mathbb{F}_2$ or $\mathbb{Z}[i]/2\mathbb{Z}[i]$.

A code C of length n over R (or an R-code of length n) is an R-submodule of R^n. An element of C is called a codeword of C. A generator matrix of C is a matrix whose rows generate C. The Hamming weight $\mathrm{wt}_H(x)$ of a codeword x is the number of non-zero components. The Lee weights of the elements $0, 1, u, 1 + u$ of $\mathbb{F}_2 + u\mathbb{F}_2$ are $0, 1, 2$ and 1, respectively, and the Lee weight $\mathrm{wt}_L(x)$ of a codeword x is the sum of the Lee weights of its components. The minimum Hamming and Lee weights d_H and d_L of C are the smallest Hamming and Lee weights among all non-zero codewords of C, respectively. We say that two codes are *equivalent* if one can be obtained from the other by permuting the coordinates and (if necessary) interchanging the two elements 1 and $1+u$ of certain coordinates for $\mathbb{F}_2+u\mathbb{F}_2$. Codes differing by only a permutation of coordinates are called *permutation-equivalent*. The automorphism group $\mathrm{Aut}(C)$ of C consists of all permutations and changes of the above two elements of the coordinates that preserve C.

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two elements of R^n. We define the inner product of x and y in R^n by $x \cdot y = x_1y_1 + \cdots + x_ny_n$. The dual code $C^\perp$ of C is defined as $C^\perp = \{x \in R^n \mid x \cdot y = 0 \text{ for all } y \in C\}$. C is *self-dual* if $C = C^\perp$. A code C is called *Type II* if $C = C^\perp$ and $\mathrm{wt}_L(x) \equiv 0 \pmod 4$ for all codewords $x \in C$. A self-dual code which are not Type II is called Type I.

2.2 The Gray Map

A map ϕ from R to $\mathbb{F}_2^2$ is defined by $\phi(0) = (0,0), \phi(1) = (0,1), \phi(u) = (1,1)$ and $\phi(1+u) = (1,0)$. The map is extended to R^n naturally. The map ϕ is an isometry from (R^n, Lee distance) to ($\mathbb{F}_2^{2n}$, Hamming distance), and called the *Gray map*. The Gray map gives several basic properties. For example, we have the following:

Proposition 2.1 (Dougherty et. al [9]). *If C is a Type II R-code of length n and minimum Lee weight d_L then $\phi(C)$ is a binary Type II $[2n, n, d_L]$ code.*

Thus if there is a Type II code over R then the length is divisible by four. In particular, there is a Type II code of length 4. Several upper bounds on minimum weights of binary Type II codes and Type I codes are known. Using a known upper bound for binary Type II codes, the minimum Lee weight $d_L(II, n)$ of a Type II code of length n is bounded by $d_L(II, n) \le 4\left[\frac{n}{12}\right] + 4$. Similarly, the minimum Lee weight $d_L(I, n)$ of a Type I code of length n is bounded by $d_L(I, n) \le 4\left[\frac{n}{12}\right] + 2$, if $n \equiv 0 \pmod{12}$, $4\left[\frac{n}{12}\right] + 6$, if $n \equiv 11 \pmod{12}$ and $4\left[\frac{n}{12}\right] + 4$ otherwise. Note that the upper bound is incorrectly reported in [9, Corollary 3.3] as $4\left[\frac{n}{24}\right] + 4$ if $n \not\equiv 22 \pmod{24}$ and $4\left[\frac{n}{24}\right] + 6$ otherwise.

2.3 Symmetrized Biweight Enumerators

We first define a relation $\sim$ on R^2 as follows: $a \sim b$ if and only if $a = b$ or $a = ib$ for any elements $a, b \in R^2$. The relation $\sim$ gives an equivalence relation on R^2. We set $\overline{R^2} = R^2/\sim$. We denote an equivalence class in $\overline{R^2}$ which contains an element $a \in R^2$ by $\overline{a}$. For example, $\overline{(1,1)} = \{(1,1), (i,i)\}$.

Let $X_{\overline{a}}$ ($\overline{a} \in \overline{R^2}$) be independent variables. Set

$$N_{\overline{a}}(x, y) = |\{j \mid \overline{(x_j, y_j)} = \overline{a}\}|,$$

where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in R^n$. The *symmetrized biweight enumerator* of a code C of length n is defined by

$$\mathrm{swe}_C(X_{\overline{a}}; \overline{a} \in \overline{R^2}) = \sum_{x,y \in C} \prod_{\overline{a} \in \overline{R^2}} X_{\overline{a}}^{N_{\overline{a}}(x,y)} \in \mathbb{Z}[X_{\overline{a}}; \overline{a} \in \overline{R^2}].$$

Often we denote $\mathrm{swe}_C(X_{\overline{a}}; \overline{a} \in \overline{R^2})$ by $\mathrm{swe}_C(X_{\overline{a}})$ shortly. This is a homogeneous polynomial of degree n in ten variables $X_{\overline{a}}$ ($\overline{a} \in \overline{R^2}$).

For several types of weight enumerators, the MacWilliams identities are known. For the symmetrized biweight enumerators, one can easily establish the following MacWilliams identity:

Lemma 2.2. *Let C be an R-code of length n. Then*

$$\mathrm{swe}_{C^\perp}(X_{\overline{a}}; \overline{a} \in \overline{R^2}) = \frac{1}{|C|^2} \mathrm{swe}_{C^\perp}\Big(\sum_{\overline{b} \in \overline{R^2}} \sum_{c \in \overline{b}} (-1)^{\mathrm{Re}({}^t ac)} X_{\overline{b}}; \overline{a} \in \overline{R^2}\Big).$$

2.4 Residue Codes and Torsion Codes

Any code over R is permutation-equivalent to a code C with generator matrix of the form

$$\begin{pmatrix} I_{k_1} & A & B_1 + uB_2 \\ 0 & uI_{k_2} & uD \end{pmatrix}, \tag{2}$$

where A, B_1, B_2 and D are matrices over $\mathbb{F}_2$. The binary $[n, k_1]$ code $C^{(1)}$ with generator matrix

$$\begin{pmatrix} I_{k_1} & A & B_1 \end{pmatrix}, \tag{3}$$

is called the *residue code* of the code C. The binary $[n, k_1 + k_2]$ code $C^{(2)}$ with generator matrix

$$\begin{pmatrix} I_{k_1} & A & B_1 \\ 0 & I_{k_2} & D \end{pmatrix}, \tag{4}$$

is called the *torsion code* of C.

We have the following characterizations for residue codes and torsion codes of self-dual codes over R.

Proposition 2.3 (Dougherty et. al [9]). *If C is a self-dual code over R then $C^{(1)}$ is a binary self-orthogonal code with ${C^{(1)}}^{\perp} = C^{(2)}$. Moreover, if C is Type II then $C^{(1)}$ contains the all-ones vector.*

Remark. For the first assertion, see the proof of Proposition 4.3 in [9].

2.5 Hermitian Modular Forms of Degree 2

Let $\mathbb{H}_2$ be the Hermitian upper half space of degree 2:

$$\mathbb{H}_2 = \{\tau \in M_2(\mathbb{C}) | (\tau - \tau^*)/2i > 0\},$$

where τ^* denotes the transpose of the complex conjugate of τ. Let Γ be the Hermitian modular group:

$$\Gamma = \{g \in GL(4, \mathbb{Z}[i]) | g^* J g = J\},$$

where

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

A holomorphic function f on $\mathbb{H}_2$ is called a *Hermitian modular form* (of degree 2) of weight n if

$$f(\tau) = f({}^t\tau) \quad \text{for all } \tau \in \mathbb{H}_2, \tag{5}$$

$$f(\tau) = \det(C\tau + D)^{-n} f((A\tau + B)(C\tau + D)^{-1}) \quad \text{for all } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma. \tag{6}$$

3 A Mass Formula of Type II Codes

In [9], a mass formula for Type II codes was given. Unfortunately, the mass formula in [9] contains some errors. Here we correct the errors and give the corrected mass formula. Note that the classifications of Type II codes of lengths 4 and 8 given in [9] are correct.

Theorem 3.1. *Let C be a code over R. Suppose that $C^{(1)}$ and $C^{(2)}$ have generator matrices given by (3) and (4), respectively. If C is Type II, then there exists a unique $(1,0)$-matrix B such that*

$$\begin{pmatrix} I_{k_1}+uB & A & B_1 \\ 0 & uI_{k_2} & uD \end{pmatrix} \tag{7}$$

is a generator matrix of C. Moreover, we have

1) *$C^{(2)} = {C^{(1)}}^{\perp}$,*
2) *The residue code $C^{(1)}$ is a self-orthogonal code containing the all-ones vector,*
3) *B is symmetric and the Lee weights of the first k_1 rows of the generator matrix are a multiple of 4.*

Conversely, if $C^{(1)}$ and $C^{(2)}$ are binary codes with generator matrices given by (3) and (4), respectively, and if the conditions 1)–3) are satisfied, then the R-code C with generator matrix (7) is a Type II code.

Proof. Since C is self-dual, Section III of [13] implies that there exists a unique matrix B such that (7) is a generator matrix of C and $C^{(2)} = {C^{(1)}}^{\perp}$. By Proposition 2.3, if C is self-dual then $C^{(1)}$ is self-orthogonal. Moreover, if C is Type II then $C^{(1)}$ contains the all-ones vector. If C is Type II then the Lee weights of the first k_1 rows of the generator matrix must be a multiple of 4. Since $(\ I_{k_1}\ A\ B_1)(\ I_{k_1}\ A\ \ B_1)^T = 0$,

$$(\ I_{k_1}+uB\ \ A\ \ B_1)(\ I_{k_1}+uB\ \ A\ \ B_1)^T = uB + uB^T = 0.$$

Thus B is symmetric.

Conversely, under the conditions 1)–3), the code C is self-dual, and each row of the matrix (7) has Lee weight divisible by four. By Proposition 4.3 in [9], each codeword of C has Lee weight divisible by four. $\square$

Therefore we have the corrected mass formula.

Theorem 3.2. *Let $N_{d_{II}}(n)$ be the number of distinct Type II R-codes of length n and let $\sigma(n,k)$ be the number of distinct binary self-orthogonal codes of length n and dimension k containing the all-ones vector, then*

$$N_{d_{II}}(n) = \sum_{k \le \frac{n}{2}} \sigma(n,k) \cdot 2^{1+\frac{k(k-1)}{2}}.$$

Proof. Theorem 3.1 implies that any Type II code is completely determined by its residue code, its torsion code and the matrix B. The number of choices for the residue code of dimension k is $\sigma(n,k)$ and the torsion code is determined uniquely by the residue code. It remains to compute the number of choices for B. We can choose freely the diagonal entries in B and the entries below the diagonal except all the entries in the first column. Since B is symmetric, the entries above the diagonal are obtained from the entries below the diagonal. The entries of the first column except the first row are determined by the condition that the Lee weights of the first k_1 rows of the generator matrix (7) are a multiple of 4. Finally, we check if the Lee weight of the first row of the generator matrix is divisible by four. Since $C^{(1)}$ contains the all-ones vector, the entries of the sum of the first k rows in (7) are 1 or $1+u$. Thus the sum of the first k rows has Lee weight divisible by four. Hence the first row has Lee weight divisible by four by Proposition 4.3 in [9]. Therefore there are $2^{k+\frac{(k-1)(k-2)}{2}}$ ways of choosing B. $\square$

Remark. $\sigma(n,k)$ is the number of self-orthogonal subspaces of dimension $k-1$ in a symplectic geometry of dimension $n-2$. The formula for $\sigma(n,k)$ is found in Ex. 8.1 of [32]

$$\sigma(n,k) = \prod_{i=0}^{k-2} \frac{2^{n-2i-2}-1}{2^{i+1}-1},$$

where $\sigma(n,0) = 0$ and $\sigma(n,1) = 1$.

Theorem 3.2 gives $N_{d_{II}}(4) = 14$ and $N_{d_{II}}(8) = 22574$. In [9], Type II codes of lengths 4 and 8 are classified, and from Table I in [9], one can check $N_{d_{II}}(n) = \sum_{C} \frac{2^n.n!}{|\operatorname{Aut}(C)|}$ for $n = 4$ and 8, where C runs through the inequivalent Type II codes of length n.

4 Relation to Binary Self-Dual Codes

In this section, we give a method for a classification of Type II codes over R using binary Type II codes.

Let A be the group of permutations of R^n generated by S_n and $m_1, \ldots, m_n$, where S_n is the coordinate permutations, m_j is defined by

$$m_j : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{j-1}, (1+u)x_j, x_{j+1}, \ldots, x_n)$$

for $(x_1, \ldots, x_n) \in R^n$. Recall that two codes C_1, C_2 over R are said to be equivalent if there exists an element $f \in A$ such that $f(C_1) = C_2$.

Let $\phi : R^n \to \mathbb{F}_2^{2n}$ be the Gray map. Then

$$\phi m_j \phi^{-1} : \mathbb{F}_2^{2n} \to \mathbb{F}_2^{2n}$$

is the coordinate permutation $(2j-1, 2j)$. If $\sigma \in S_n$ is a coordinate permutation of R^n, then $\tilde{\sigma} = \phi\sigma\phi^{-1}$ is the coordinate permutation of $\mathbb{F}_2^{2n}$ defined by

$$\begin{aligned} \tilde{\sigma}(2j-1) &= 2\sigma(j) - 1, \\ \tilde{\sigma}(2j) &= 2\sigma(j) \end{aligned}$$

for $j = 1, 2, \ldots, n$. Now it is clear that $\phi^{-1}A\phi$ coincides with the centralizer $C_{S_{2n}}(\tau)$ of the element $\tau = (1,2)(3,4)\cdots(2n-1,2n)$.

Proposition 4.1. *Let C_1, C_2 be codes over R. Then C_1 is equivalent to C_2 if and only if there exists an element $\rho \in C_{S_{2n}}(\tau)$ such that $\rho(\phi(C_1)) = \phi(C_2)$.*

Proof. If C_1 is equivalent to C_2, then there exists an element $f \in A$ such that $f(C_1) = C_2$. Taking $\rho = \phi f \phi^{-1}$, we obtain $\rho(\phi(C_1)) = \phi(C_2)$. Conversely, if $\rho(\phi(C_1)) = \phi(C_2)$ for some $\rho \in C_{S_{2n}}(\tau)$, then defining f by $f = \phi^{-1}\rho\phi$ we find $f \in A$ and $f(C_1) = C_2$. $\square$

Recall that the automorphism group of an R-code C is by definition

$$\mathrm{Aut}(C) = \{f \in A | f(C) = C\}.$$

Proposition 4.2. *Let C be a code over R. Then* $\mathrm{Aut}(C) \cong C_{\mathrm{Aut}(\phi(C))}(\tau)$.

Proof. We have

$$\begin{aligned} \mathrm{Aut}(C) &\cong \phi(\mathrm{Aut}(C))\phi^{-1} \\ &= \{\phi f \phi^{-1} | f \in A,\ f(C) = C\} \\ &= \{\rho \in C_{S_{2n}}(\tau) | \rho(\phi(C)) = \phi(C)\} \\ &= C_{\mathrm{Aut}(\phi(C))}(\tau), \end{aligned}$$

as desired. $\square$

Let $\mathcal{D}$ be the set of pairs (D, τ') where D is a binary code of length $2n$, $\tau' \in \mathrm{Aut}(D)$ is a fixed-point-free involution. We define an equivalence relation on $\mathcal{D}$ as follows; two elements $(D_1, \tau_1), (D_2, \tau_2)$ are equivalent if and only if there exists an element $\rho \in S_{2n}$ such that $\rho(D_1) = D_2$ and $\rho\tau_1\rho^{-1} = \tau_2$. The equivalence class containing (D, τ') is denoted by $[D, \tau']$. We denote by $\bar{\mathcal{D}}$ the set of equivalence classes.

Let $\mathcal{C}$ be the set of codes over R of length n. For each $C \in \mathcal{C}$, let $[C]$ denote the equivalence class containing C. We denote by $\bar{\mathcal{C}}$ the set of equivalence classes of codes over R of length n.

Proposition 4.3. *There is a one-to-one correspondence between $\bar{\mathcal{C}}$ and $\bar{\mathcal{D}}$ given by*

$$[C] \mapsto [\phi(C), \tau]. \tag{8}$$

Proof. We see readily from Proposition 4.1 that the mapping (8) is well-defined and injective. To show that (8) is surjective, pick $[D, \tau'] \in \bar{\mathcal{D}}$. Then there exists an element $\sigma \in S_{2n}$ such that $\sigma\tau'\sigma^{-1} = \tau$, hence $[D, \tau'] = [\sigma(D), \tau]$. Now $\phi^{-1}(\sigma(D))$ is a code over R of length $2n$. Thus (8) is surjective. $\square$

By Proposition 4.3, the classification of codes over R of length n reduces to the classification of binary codes of length $2n$ and the classification of conjugacy classes of fixed-point-free involutions in the automorphism group of each of the binary codes of length $2n$. The automorphism group of a code over R can be computed from the automorphism group of the corresponding binary code by Proposition 4.2. Moreover, by Proposition 2.1, the classification of Type II codes over R of length n reduces to the classification of binary Type II codes of length $2n$ and the classification of conjugacy classes of fixed-point-free involutions in the automorphism group of each of the binary codes of length $2n$.

5 Classification of Type II Codes of Lengths 12 and 16

A classification of Type II codes of lengths 4 and 8 was given in [9]. This was done directly. In this section, we extend the classification to lengths 12 and 16 by the method given in the previous section.

Table 1: Classification of Type II codes of length 12

Binary Codes	Total	A_{24}	B_{24}	C_{24}	D_{24}	E_{24}	F_{24}	EE_{24}	DE_{24}	G_{24}
Numbers	82	16	3	24	4	8	8	6	12	1

5.1 Length 12

The classification of binary Type II codes of length 24 was given in [24]. There are exactly nine inequivalent binary Type II codes, seven of which are indecomposable. The seven codes are denoted by $A_{24}, B_{24}, \ldots, G_{24}$ in [24] and we use these notations. We denote the remaining two decomposable codes by EE_{24} and DE_{24} where EE_{24} is the code which is the direct sum of three copies of the Hamming $[8, 4, 4]$ code. As described above, our method is to obtain the automorphism group of a binary Type II code then to obtain conjugacy classes of fixed-point-free involutions in the automorphism group. We complete the classification of Type II codes of length 12 by listing the number of inequivalent codes in Table 1 and giving their generator matrices in Appendix I. In the table, each number denotes the inequivalent Type II codes obtained from the binary code given in the first row.

By Proposition 4.2, the orders of the automorphism groups of the inequivalent Type II codes are computed. Then we have $N_{d_{II}}(12) = 6667691054 = \sum_C \frac{2^{12}.12!}{|\operatorname{Aut}(C)|}$ where C runs through the inequivalent Type II codes, showing that our classification is complete.

Proposition 5.1. *There are exactly* 82 *inequivalent Type II codes of length* 12*. There is a unique Type II code with minimum Lee weight* 8 *of length* 12*. There are exactly four inequivalent Type II codes minimum Hamming weight* 4 *of length* 12*.*

We now give some observation of the Type II codes of length 12. Since the Gray map ϕ is an isometry map from (R^n, Lee weight) to (F^{2n}, Hamming weight), the minimum Lee weight of a Type II code C over R is the same as the minimum Hamming weight of $\phi(C)$. It is well known that the Golay code G_{24} is the unique binary Type II $[24, 12, 8]$ code. By the classification, only one Type II code over R is constructed from G_{24}. Thus $G_{12,1}$ is a unique Type II code with minimum Lee weight 8. Only $A_{12,16}$, $D_{12,2}$, $F_{12,8}$ and $G_{12,1}$ have minimum Hamming weight 4 and the others have minimum Hamming weight 2. The symmetrized weight enumerator of the unique Type II code $G_{12,1}$ with minimum Lee weight 8 is

$$\begin{aligned}
\mathrm{swe}_{G_{12,1}}(a,b,c) \quad = \quad & a^{12} + 15a^8c^4 + 240a^6b^4c^2 + 32a^6c^6 + 384a^5b^6c \\
& +120a^4b^8 + 480a^4b^4c^4 + 15a^4c^8 + 1280a^3b^6c^3 + 720a^2b^8c^2 \\
& +240a^2b^4c^6 + 384ab^6c^5 + 64b^{12} + 120b^8c^4 + c^{12}.
\end{aligned}$$

5.2 Length 16

Similarly to length 12, we complete the classification of Type II codes of length 16 noting that there are exactly 85 the binary Type II codes of length 32 [6] (see also [7]). However we only list in Table 2 how many Type II codes are constructed from each binary code, since there are a large number of such codes. For the 85 binary Type II codes, we use the notations given in [7]. From Table 2, we have the following classification. We have checked the mass formula $N_{d_{II}}(16) = 461203898158916654 = \sum_C \frac{2^{16}.16!}{|\operatorname{Aut}(C)|}$ where C runs through the inequivalent Type II codes.

Proposition 5.2. *There are exactly* 1894 *inequivalent Type II codes of length* 16*,* 21 *of which have minimum Lee weight* 8*.*

The 21 Type II codes with minimum Lee weight 8 have minimum Hamming weight 4. For the 21 Type II codes, we give generator matrices in Appendix II. Generator matrices of the other codes are available from the authors.

Table 2: Type II codes of length 16

Codes	Numbers	Codes	Numbers	Codes	Numbers	Codes	Numbers
C1	10	C23	3	C45	2	C67	24
C2	16	C24	9	C46	0	C68	38
C3	35	C25	48	C47	0	C69	52
C4	5	C26	8	C48	0	C70	26
C5	22	C27	16	C49	0	C71	27
C6	24	C28	2	C50	2	C72	4
C7	66	C29	46	C51	13	C73	3
C8	0	C30	67	C52	5	C74	43
C9	8	C31	124	C53	10	C75	24
C10	32	C32	11	C54	6	C76	2
C11	64	C33	14	C55	2	C77	22
C12	165	C34	156	C56	4	C78	14
C13	10	C35	0	C57	39	C79	4
C14	10	C36	20	C58	2	C80	7
C15	42	C37	24	C59	13	C81	1
C16	3	C38	40	C60	8	C82	3
C17	10	C39	14	C61	26	C83	6
C18	6	C40	200	C62	2	C84	4
C19	0	C41	24	C63	16	C85	7
C20	12	C42	22	C64	2		
C21	12	C43	12	C65	6		
C22	9	C44	2	C66	2	Total	1894

6 Construction of Hermitian Modular Forms

For a polynomial $f(X_{\overline{a}}) \in \mathbb{C}[X_{\overline{a}}; \overline{a} \in \overline{R^2}]$ and a matrix T, we define

$$f^T(X_{\overline{a}}) = f(\sum_{\overline{b}\in\overline{R^2}} t_{\overline{a},\overline{b}} X_{\overline{b}}),$$

where $T = (t_{\overline{a},\overline{b}})_{\overline{a},\overline{b}\in\overline{R^2}}$. Set

$$T = -\frac{1}{4}\left(\sum_{c\in\overline{b}} (-1)^{\mathrm{Re}({}^t a\, c)}\right)_{\overline{a},\overline{b}\in\overline{R^2}}. \tag{9}$$

Since $\mathrm{Re}({}^t(ia)c) = \mathrm{Re}({}^t a(ic))$ and $\overline{b} = \{b, ib\}$, the right hand side of (9) is well-defined. Suppose that C is a self-dual code of length n over R. Since $|C| = 2^n$, Lemma 2.2 implies

$$\mathrm{swe}_C^T(X_{\overline{a}}) = \mathrm{swe}_{C^\perp}(X_{\overline{a}}) = \mathrm{swe}_C(X_{\overline{a}}).$$

We define the mapping $A \mapsto P_A = (\delta_{\overline{A^* a}, \overline{b}})_{\overline{a}, \overline{b} \in \overline{R^2}}$, which gives a homomorphism from $GL(2, \mathbb{Z}[i])$ to $GL(10, \mathbb{Z})$. Now

$$\mathrm{swe}_C^{P_A}(X_{\overline{a}}) = \mathrm{swe}_C(X_{\overline{A^* a}}) = \sum_{c_1, c_2 \in C} \prod_{\overline{a} \in \overline{R^2}} X_{\overline{a}}^{n_{\overline{(A^*)^{-1} a}}(c_1, c_2)}.$$

If

$$A^* = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix},$$

then

$$\begin{aligned} n_{\overline{(A^*)^{-1} a}}(c_1, c_2) &= |\{j | A^{*t}(c_{1j}, c_{2j}) \in \overline{a}\}| \\ &= n_{\overline{a}}(\alpha_{11} c_1 + \alpha_{12} c_2, \alpha_{21} c_1 + \alpha_{22} c_2). \end{aligned}$$

Since

$$(c_1, c_2) \mapsto (\alpha_{11} c_1 + \alpha_{12} c_2, \alpha_{21} c_1 + \alpha_{22} c_2)$$

is a bijection on $C \times C$, we see that

$$\mathrm{swe}_C^{P_A}(X_{\overline{a}}) = \sum_{c_1, c_2 \in C} \prod_{\overline{a} \in \overline{R^2}} X_{\overline{a}}^{n_{\overline{a}}(\alpha_{11} c_1 + \alpha_{12} c_2, \alpha_{21} c_1 + \alpha_{22} c_2)} = \mathrm{swe}_C(X_{\overline{a}}).$$

For a matrix $S = S^* \in \mathrm{Mat}(2, \mathbb{Z}[i])$, we define a mapping $\overline{R^2} \to \mathbb{Z}/4\mathbb{Z}$, $\overline{a} \mapsto S[a]$ as follows. For each $a \in R^2$, regard a as an element of $\mathbb{Z}[i]^2$ and define $S[a] = a^* S a$, where a^* is the conjugate transpose of a. Since $(a^* S a)^* = a^* S a$, $(ia)^* S(ia) = (-i)i(a^* S a) = a^* S a$, and

$$(a + 2b)^* S(a + 2b) = a^* S a + 4(\mathrm{Re}(a^* S b) + b^* S b) \equiv a^* S a \pmod 4,$$

the mapping $\overline{a} \mapsto S[a]$ is well-defined as a mapping from $\overline{R^2}$ to $\mathbb{Z}/4\mathbb{Z}$. The values of $S[a] \in \mathbb{Z}/4\mathbb{Z}$ ($\overline{a} \in \overline{R^2}$) for

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{10}$$

are given in Table 3, where $u = 1 + i$.

It follows from Table 3 that for $c_1, c_2 \in C$,

$$\sum_{\overline{a} \in \overline{R^2}} S[a] n_{\overline{a}}(c_1, c_2) \equiv \begin{cases} \mathrm{wt}_L(c_1) \pmod 4 & \text{if } S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ \mathrm{wt}_L(c_2) \pmod 4 & \text{if } S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ 2\,\mathrm{Re}((c_1, c_2)) \pmod 4 & \text{if } S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ 2\,\mathrm{Re}(i(c_1, c_2)) \pmod 4 & \text{if } S = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \end{cases}$$

Table 3: The values of $S[a]$

	$\overline{\begin{pmatrix}0\\0\end{pmatrix}}$	$\overline{\begin{pmatrix}0\\1\end{pmatrix}}$	$\overline{\begin{pmatrix}0\\u\end{pmatrix}}$	$\overline{\begin{pmatrix}1\\0\end{pmatrix}}$	$\overline{\begin{pmatrix}u\\0\end{pmatrix}}$	$\overline{\begin{pmatrix}1\\1\end{pmatrix}}$	$\overline{\begin{pmatrix}1\\i\end{pmatrix}}$	$\overline{\begin{pmatrix}1\\u\end{pmatrix}}$	$\overline{\begin{pmatrix}u\\1\end{pmatrix}}$	$\overline{\begin{pmatrix}u\\u\end{pmatrix}}$
$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	0	0	0	1	2	1	1	1	2	2
$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	0	1	2	0	0	1	1	2	1	2
$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	0	0	0	0	0	2	0	2	2	0
$\begin{pmatrix}0&i\\-i&0\end{pmatrix}$	0	0	0	0	0	0	2	2	2	0

Thus, if C is Type II, then for each S in (10), we have

$$\prod_{\overline{a}\in\overline{R^2}} i^{S[a]n_{\overline{a}}(c_1,c_2)} = 1.$$

Therefore, if we let D_S be the diagonal matrix with $(\overline{a},\overline{a})$-entry $i^{S[a]}$, then

$$\mathrm{swe}_C^{D_S}(X_{\overline{a}}) = \sum_{c_1,c_2\in C} \prod_{\overline{a}\in\overline{R^2}} i^{S[a]n_{\overline{a}}(c_1,c_2)} \prod_{\overline{a}\in\overline{R^2}} X_{\overline{a}}^{n_{\overline{a}}(c_1,c_2)} = \mathrm{swe}_C(X_{\overline{a}}).$$

Since $D_{S_1+S_2} = D_{S_1}D_{S_2}$ for any $S_1 = S_1^*, S_2 = S_2^* \in \mathrm{Mat}(2,\mathbb{Z}[i])$, we see that $\mathrm{swe}_C(X_{\overline{a}})$ is invariant under D_S for all $S = S^* \in \mathrm{Mat}(2,\mathbb{Z}[i])$.

Finally, as n is divisible by 4, $\mathrm{swe}_C(X_{\overline{a}})$ is invariant under iI. We have proved the following.

Proposition 6.1. *For any Type II code C over R, the symmetrized biweight enumerator $\mathrm{swe}_C(X_{\overline{a}})$ is invariant under the action of the matrices T, P_A, D_S, iI, where $A \in GL(2,\mathbb{Z}[i])$ and $S = S^* \in \mathrm{Mat}(2,\mathbb{Z}[i])$.*

We now define the theta functions $\Theta\begin{bmatrix}a\\b\end{bmatrix}(\tau)$ on $\mathbb{H}_2$ with characteristic $\begin{bmatrix}a\\b\end{bmatrix}$, a, $b \in \mathbb{Z}[i]^2$ as follows:

$$\Theta\begin{bmatrix}a\\b\end{bmatrix}(\tau) = \sum_{n\in\mathbb{Z}[i]^2} \mathbf{e}\left(\frac{1}{2}(n+\frac{1}{2}a)^*\tau(n+\frac{1}{2}a) + \frac{1}{2}\mathrm{Re}(b^*n)\right),$$

where $\tau \in \mathbb{H}_2$, $\mathbf{e}(x) = \exp(2\pi i x)$. The following basic properties are used in this paper.

Lemma 6.2. (i) $\Theta\begin{bmatrix}ia\\ib\end{bmatrix}(\tau) = \Theta\begin{bmatrix}a\\b\end{bmatrix}(\tau)$.

(ii) $\Theta\left[\begin{array}{c} a+2r \\ b+2s \end{array}\right](\tau) = \mathbf{e}\left(\frac{1}{2}\operatorname{Re}(b^* r)\right)\Theta\left[\begin{array}{c} a \\ b \end{array}\right](\tau)$, *where* $r, s \in \mathbb{Z}[i]$.

(iii) $\Theta\left[\begin{array}{c} a \\ b \end{array}\right]({}^t\tau) = \mathbf{e}\left(-\frac{1}{2}\operatorname{Re}({}^t a)\cdot\operatorname{Re}(b)\right)\Theta\left[\begin{array}{c} a \\ b \end{array}\right](\tau)$.

(iv) $\Theta\left[\begin{array}{c} a \\ b \end{array}\right](J\cdot\tau) = -\mathbf{e}\left(\frac{1}{4}\operatorname{Re}(a^* b)\right)\det(\tau)\Theta\left[\begin{array}{c} b \\ a \end{array}\right](\tau)$.

(v) *If* g *is of the form* $\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$, $A \in GL(2, \mathbb{Z}[i])$, *then*

$$\Theta\left[\begin{array}{c} a \\ b \end{array}\right](g\cdot\tau) = \Theta\left[\begin{array}{c} A^* a \\ A^{-1} b \end{array}\right](\tau)$$

(vi) *If* g *is of the form* $\begin{pmatrix} I_2 & S \\ 0 & I_2 \end{pmatrix}$, $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = S^* \in \operatorname{Mat}(2, \mathbb{Z}[i])$, *then*

$$\Theta\left[\begin{array}{c} a \\ b \end{array}\right](g\cdot\tau) = \mathbf{e}\left(\frac{1}{8}a^*\, Sa\right)\Theta\left[\begin{array}{c} a \\ \tilde{b} \end{array}\right](\tau),$$

where $\tilde{b} = b + Sa + (1-i)\begin{pmatrix} S_{11} \\ S_{22} \end{pmatrix}$.

Proof. These formulae can be derived from Lemma 2.1.1 (i), (ii), Lemma 2.1.2 (i), (x), (viii), (ix) in [20] respectivly. See also pp. 7–9 in [12], Chap. IV, Sections 1 and 2 in [19]. □

By Lemma 6.2 (i), (ii), we see that the function $\Theta\begin{bmatrix} a \\ 0 \end{bmatrix}(\tau)$ depends only on the class of a in $\overline{R^2}$. Thus we may define $f_{\overline{a}}(\tau) = \Theta\begin{bmatrix} a \\ 0 \end{bmatrix}(2\tau)$.

Lemma 6.3. (i) $f_{\overline{a}}({}^t\tau) = f_{\overline{a}}(\tau)$.

(ii) $f_{\overline{a}}(J\cdot\tau) = \det(\tau)\sum_{\overline{b}\in\overline{R^2}} T_{\overline{a},\overline{b}} f_{\overline{b}}(\tau)$.

(iii) *If* g *is of the form* $\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$, $A \in GL(2, \mathbb{Z}[i])$, *then* $f_{\overline{a}}(g\cdot\tau) = f_{\overline{A^* a}}(\tau)$.

(iv) *If* g *is of the form* $\begin{pmatrix} I_2 & S \\ 0 & I_2 \end{pmatrix}$, $S = S^* \in \operatorname{Mat}(2, \mathbb{Z}[i])$, *then* $f_{\overline{a}}(g\cdot\tau) = i^{S[a]} f_{\overline{a}}(\tau)$.

Proof. (i) Follows from Lemma 6.2 (iii).
(ii) By Lemma 6.2 (iv), we have

$$\begin{aligned}
f_{\overline{a}}(J\cdot\tau) &= \Theta\begin{bmatrix}a\\0\end{bmatrix}(J\cdot(\frac{1}{2}\tau))\\
&= -\det(\frac{1}{2}\tau)\Theta\begin{bmatrix}0\\a\end{bmatrix}(\frac{1}{2}\tau)\\
&= -\frac{1}{4}\det(\tau)\sum_{b\in R^2}\sum_{m\in\mathbb{Z}[i]^2}\mathbf{e}(\frac{1}{4}(b+2m)^*\tau(b+2m)+\frac{1}{2}\operatorname{Re}(a^*(b+2m)))\\
&= -\frac{1}{4}\det(\tau)\sum_{b\in R^2}\mathbf{e}(\frac{1}{2}\operatorname{Re}(a^*b))\sum_{m\in\mathbb{Z}[i]^2}\mathbf{e}(\frac{1}{2}(m+\frac{1}{2}b)^*(2\tau)(m+\frac{1}{2}b))\\
&= -\frac{1}{4}\det(\tau)\sum_{b\in R^2}(-1)^{\operatorname{Re}({}^t a\, b)}f_{\overline{b}}(\tau)\\
&= \det(\tau)\sum_{\overline{b}\in\overline{R^2}}T_{\overline{a},\overline{b}}f_{\overline{b}}(\tau).
\end{aligned}$$

(iii) Follows from Lemma 6.2 (v).
(iv) By Lemma 6.2 (vi), we have

$$f_{\overline{a}}(g\cdot\tau)=\Theta\begin{bmatrix}a\\0\end{bmatrix}(2\tau+2S)=\Theta\begin{bmatrix}a\\0\end{bmatrix}(\begin{pmatrix}I_2 & 2S\\0 & I_2\end{pmatrix}\cdot(2\tau))=\mathbf{e}(\frac{1}{8}a^*(2S)a)\Theta\begin{bmatrix}a\\\tilde{b}\end{bmatrix}(2\tau),$$

where

$$\tilde{b}=2Sa+(1-i)\begin{pmatrix}2S_{11}\\2S_{22}\end{pmatrix}\equiv 0 \pmod 2.$$

Thus by Lemma 6.2 (ii), we have

$$f_{\overline{a}}(g\cdot\tau)=\mathbf{e}(\frac{1}{4}a^*Sa)\Theta\begin{bmatrix}a\\0\end{bmatrix}(2\tau)=i^{S[a]}f_{\overline{a}}(\tau).$$

□

Let G be the subgroup of $GL(10,\mathbb{Q}[i])$ generated by the matrices

$$T,\quad (\det A^*)P_A\quad (A\in GL(2,\mathbb{Z}[i])), D_S\quad (S=S^*\in \operatorname{Mat}(2,\mathbb{Z}[i])).$$

Theorem 6.4. *If $W(X_{\overline{a}};\overline{a}\in\overline{R^2})$ is a polynomial of degree n invariant under the group G, then $W(f_{\overline{a}}(\tau);\overline{a}\in\overline{R^2})$ is a Hermitian modular form of weight n for Γ.*

Proof. By Lemma 6.3 (i), the function $f(\tau)=W(f_{\overline{a}}(\tau))$ satisfies (5). By [19, Chap. II, Theorem 2.3], it is enough to check the transformation formula (6) for

$$g=J,\begin{pmatrix}A & 0\\0 & (A^*)^{-1}\end{pmatrix},\begin{pmatrix}I_2 & S\\0 & I_2\end{pmatrix},$$

where $A \in GL(2, \mathbb{Z}[i])$, $S = S^* \in \mathrm{Mat}(2, \mathbb{Z}[i])$.

For the first and third cases (of g), the assertions are easily obtained by Lemma 6.3 (ii), (iv), respectively.

For $g = \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$ with $A \in GL(2, \mathbb{Z}[i])$, we have, by Lemma 6.3 (iii),

$$\begin{aligned} f(g \cdot \tau) &= W(f_{\overline{A^* a}}(\tau)) \\ &= W^{P_A}(f_{\overline{a}}(\tau)) \\ &= (\det(A^*))^{-n}(\det(A^*))^n W^{P_A}(f_{\overline{a}}(\tau)) \\ &= (\det(A^*))^{-n} W^{(\det(A^*))P_A}(f_{\overline{a}}(\tau)) \\ &= (\det(A^*)^{-1})^n f(\tau). \end{aligned}$$

This completes the proof. $\square$

We remark that the above group G has order 737280 and Molien series

$$\begin{aligned} &\frac{1}{(1-t^{20})(1-t^{10})(1-t^{12})^3(1-t^8)^3(1-t^4)^2} \\ &\quad \times (1 + 2t^8 + 10t^{12} + 24t^{16} + 53t^{20} + 108t^{24} + 192t^{28} + 302t^{32} + 420t^{36} + 506t^{40} \\ &\qquad + 540t^{44} + 527t^{48} + 452t^{52} + 330t^{56} + 206t^{60} + 108t^{64} + 45t^{68} + 12t^{72} + 2t^{76}) \\ &= 1 + 2t^4 + 8t^8 + t^{10} + 27t^{12} + 2t^{14} + 82t^{16} + \cdots . \end{aligned}$$

Corollary 6.5. *For a Type II code C of the length n, $\mathrm{swe}_C(f_{\overline{a}}(\tau); \overline{a} \in \overline{R^2})$ is a Hermitian modular form of weight n for Γ.*

Proof. In view of Proposition 6.1 and Theorem 6.4, it suffices to show that $\mathrm{swe}_C(X_{\overline{a}}; \overline{a} \in \overline{R^2})$ is invariant under the scalar matrices $(\det A^*)I_{10}$, where $A \in GL(2, \mathbb{Z}[i])$. This is obvious since $n \equiv 0 \pmod 4$ and $\det A^* \in \{\pm 1, \pm i\}$ which imply $(\det A^*)^n = 1$. $\square$

7 The Graded Ring of Hermitian Modular Forms and Dimensions

We review here several known facts on the graded rings and add some remarks. Although the content of this section consists of rather easy comments on the previously known facts, there are several non-trivial points which have never been stated in the literature before.

For $m = (m', m'') \in \mathbb{Z}^4$, we put

$$\theta_m(\tau) = \Theta \left[\begin{array}{c} (1+i)m' \\ (1+i)m'' \end{array} \right] (\tau).$$

These theta constants θ_m were used in Freitag and Nagaoka instead of our f_a. It is known that $\theta_m(\tau)$ does not vanish identically if and only if m is even, i.e. ${}^t m' m'' \equiv 0 \pmod 2$. First we explain relations between θ_m and f_a.

Lemma 7.1. *If we put* $M = \{(0,0),(0,1),(1,0),(1,1)\}$, *we have*

$$\theta_m(\tau) = \sum_{r\in M} f_{r+i(r+m')}(\tau)e(\frac{{}^t rm''}{2}).$$

Proof. Put $\Lambda_1 = \mathbb{Z}[i]^2$, $\Lambda_2 = (1-i)\mathbb{Z}[i]^2$. Then, $\Lambda_1/\Lambda_2 = M$, identifying M as a complete set of representatives of $(\mathbb{Z}/2\mathbb{Z})^2$. Hence for any function h on Λ_1, we get

$$\sum_{n\in\Lambda_1} h(n) = \sum_{r\in M}\sum_{n_2\in\Lambda_2} h(n_2+r)$$

as in Igusa [18]. Now for $m = {}^t(m', m'') \in \mathbb{Z}^4$, put

$$h_m(x) = e(\frac{1}{2}(x+\frac{1+i}{2}m')^*\tau(x+\frac{1+i}{2}m') + Re(\frac{1+i}{2}{}^t xm'')).$$

Then, $n_2 = (1-i)x \in \Lambda_2$ and $r \in M$, we get

$$\begin{aligned} & \frac{1}{2}(n_2+r+\frac{1+i}{2}m')^*\tau(n_2+r+\frac{1+i}{2}m') + Re(\frac{1+i}{2}\,{}^t(n_2+r)m'') \\ = \;& (x+\frac{r+i(r+m')}{2})^*\tau(x+\frac{r+i(r+m')}{2}) + Re({}^t xm'' + \frac{1+i}{2}\,{}^t rm''). \end{aligned}$$

Hence taking the sum over Λ_2 and M, we get the relation of the lemma. □

More concretely, we get

$$\begin{aligned}
\theta_{0000} &= f_{0,0}+f_{0,u}+f_{u,0}+f_{u,u},\\
\theta_{0001} &= f_{0,0}-f_{0,u}+f_{u,0}-f_{u,u},\\
\theta_{0010} &= f_{0,0}+f_{0,u}-f_{u,0}-f_{u,u},\\
\theta_{0011} &= f_{0,0}-f_{0,u}-f_{u,0}+f_{u,u},\\
\theta_{0100} &= 2(f_{0,1}+f_{u,1}),\\
\theta_{0110} &= 2(f_{0,1}-f_{u,1}),\\
\theta_{1000} &= 2(f_{1,0}+f_{1,u}),\\
\theta_{1001} &= 2(f_{1,0}-f_{1,u}),\\
\theta_{1100} &= 2(f_{1,1}+f_{1,i}),\\
\theta_{1111} &= 2(f_{1,1}-f_{1,i}).
\end{aligned}$$

Nagaoka used the following notation.

$$\begin{aligned}
\psi_{4k} &= \frac{1}{4}\sum_{m:even} \theta_m^{4k} \text{ for any } k \text{ with } k \equiv 0 \pmod 4, \\
\chi_8 &= (\psi_4^2 - \psi_8)/3072, \\
\chi_{10} &= 2^{-12}\prod_{m:even} \theta_m, \\
\chi_{12} &= 2^{-15}\sum_{fifteen} (\theta_{m_1}\theta_{m_2}\theta_{m_3}\theta_{m_4}\theta_{m_5}\theta_{m_6})^2, \\
\xi_{12} &= 11\psi_4^3 - 13824\psi_4\chi_8 + 608256\chi_{12} - 9\psi_{12}, \\
\chi_{16} &= 2^{-18}\sum_{fifteen} (\theta_{m_1}\theta_{m_2}\theta_{m_3}\theta_{m_4})^4,
\end{aligned}$$

where the product for χ_{10} is over the ten even characteristics, the summation for χ_{12} is over the fifteen complements of syzygous quadruples and for χ_{16} over the fifteen azygous quadruples. We denote by $A(\Gamma)^s$ the ring of symmetric Hermitian modular forms.

Theorem 7.2 (Freitag [12], Nagaoka [21]). *$A(\Gamma)^s$ is generated by 6 modular forms ψ_4, χ_8, χ_{10}, χ_{12}, ξ_{12}, χ_{16}. These generators satisfy the following relation.*

$$2(\psi_4^2\chi_8 + 6\psi_4\chi_{12} + 4032\chi_8^2 - 72\chi_{16})^2 = (\psi_4\chi_8^2 + 12\chi_8\chi_{12} + 36\chi_{10}^2)\xi_{12}.$$

The relation above is the unique fundamental relation between generators, although this fact is mentioned just as a conjecture in Nagaoka's paper above. We give two different proofs of this fact here. The first one is shorter and smart which we learned from Freitag. Since the polynomial P of 6 variables which gives the above relation is irreducible, the ideal generated by P is a prime ideal. Since the dimension of the variety $Proj A(\Gamma)^s$ is 4, the transcendental degree of the graded ring $A(\Gamma)^s$ must be 5. But there are only 6 generators, so the height of the ideal $\mathfrak{p}$ of the whole relation of generators must be one. It is trivial that the ideal $\mathfrak{p}$ is a prime ideal containing (P). If $(P) \neq \mathfrak{p}$, then the height of $\mathfrak{p}$ is more than one by definition, which is a contradiction. Hence all the relations come from P.

We give here another proof directly obtained from the definition without using the fact that the ring is generated by the above 6 generators. For the proof, we need some results on Siegel modular forms. We denote by S_2 the Siegel upper half space of degree two and we regard this space as a subset of $\mathbb{H}_2$ naturally. For any $m \in \mathbb{Z}^4$, we define theta constants $\vartheta_m(Z)$ of characteristic m on S_2 by

$$\vartheta_m(Z) = \sum_{p \in \mathbb{Z}^2} e(\frac{1}{2}\,{}^t(p+\frac{a}{2})Z(p+\frac{a}{2}) + {}^t(p+\frac{a}{2})\frac{b}{2}),$$

where $Z \in S_2$. We denote by G_4 or G_6 the normalized Siegel Eisenstein series of weight 4 or 6, respectively. We define two more Siegel modular forms

$$\begin{aligned} X_{10} &= 2^{-12} \prod_{m:even} \vartheta_m^2, \\ X_{12} &= 2^{-15} \prod_{fifteen} (\vartheta_{m_1}\vartheta_{m_2}\vartheta_{m_3}\vartheta_{m_4}\vartheta_{m_5}\vartheta_{m_6})^4, \end{aligned}$$

where the first product is taken over 10 even characteristics and the second one over the set of complements of fifteen syzygous quadruples. It is well known that these 4 forms G_4, G_6, X_{10}, X_{12} are algebraically independent and generate the ring of Siegel modular forms of even weights (cf. Igusa [17]).

For a Hermitian modular form $F(\tau)$ of any Γ', we write the restriction of F to S_2 by $F|S_2$. Since $Sp(2,\mathbb{Z}) \subset \Gamma$, the restriction $F|S_2$ of any $F \in A_k(\Gamma)^s$ is a Siegel modular form of $Sp(2,\mathbb{Z})$ of weight k. Nagaoka has shown that

$$\psi_4|S_2 = G_4, \qquad \chi_8|S_2 = 0, \qquad \chi_{10}|S_2 = X_{10}, \qquad \chi_{12}|S_2 = X_{12}$$

$$\xi_{12}|S_2 = 2G_6^2 \qquad \chi_{16}|S_2 = 2^{-3}3^{-1}(G_4X_{12} - G_6X_{10}).$$

Let P or Q be a polynomial of four or five variables respectively. Assume that

$$P(\psi_4, \chi_{10}, \chi_{12}, \xi_{12}) + \chi_8 Q(\psi_4, \chi_8, \chi_{10}, \chi_{12}, \xi_{12}) = 0.$$

By restricting this relation to S_2, we get $P(G_4, X_{10}, X_{12}, 2G_6^2) = 0$. But G_4, X_{10}, X_{12}, G_6 are algebraically independent. Hence $P = 0$. So, we get
$Q(\psi_4, \chi_8, \chi_{10}, \chi_{12}, \xi_{12}) = 0$. By induction, we see $Q = 0$ and prove the algebraic independence of five forms. Now, by Nagaoka's relation, we see that χ_{16}^2 is in $A + \chi_{16}A$. So, all we should do is to show that this is a direct sum. Again let P and Q be polynomials of five variables. Assume that

$$P(\psi_4, \chi_8, \chi_{10}, \chi_{12}, \xi_{12}) + \chi_{16} Q(\psi_4, \chi_8, \chi_{10}, \chi_{12}, \xi_{12}) = 0.$$

If we restrict this to S_2, then we get

$$P(G_4, 0, X_{10}, X_{12}, 2G_6^2) + 2^{-3}3^{-1}(G_4X_{12} - G_6X_{10})Q(G_4, 0, X_{10}, X_{12}, 2G_6^2) = 0.$$

Since odd powers of G_6 appears as $-2^{-3}3^{-1}G_6X_{10}Q(G_4, 0, X_{10}, X_{12}, 2G_6^2)$, this term should be 0. So we get

$$P(G_4, 0, X_{10}, X_{12}, 2G_6^2) = Q(G_4, 0, X_{10}, X_{12}, 2G_6^2) = 0.$$

So both P and Q are divisible by the second variable, and we can divide the relation by χ_8. We can continue this process and by induction, we see that $P = Q = 0$. Hence the alternative proof is finished.

The above theorem by Freitag and Nagaoka is described also as follows. The five forms ψ_4, χ_8, χ_{10}, χ_{12}, ξ_{12} are algebraically independent. If we put $A = \mathbb{C}[\psi_4, \chi_8, \chi_{10}, \chi_{12}, \xi_{12}]$, we get

$$A(\Gamma)^s = A \oplus \chi_{16} A,$$

where $\oplus$ means the direct sum as modules. By the way, $\chi_8 = (\psi_4^2 - \psi_8)/3072$ and χ_{16} is a linear conbination of ψ_{16}, $\psi_4\psi_{12}$, ψ_4^3, $\psi_4^2\chi_8$, $\psi_4\chi_{12}$, χ_8^2 in which the coefficient of ψ_{16} does not vanish (cf. Nagaoka [21] p. 537). So, $A(\Gamma)^s$ is also generated by ψ_4, ψ_8, ψ_{12}, χ_{12}, ψ_{16}, χ_{10}. Nagaoka has shown that χ_8, χ_{10}, χ_{12}, χ_{16} are cusp forms. Since the image on the boundary of ψ_4 and ξ_{12} are algebraically independent, we see that the ideal of cusp forms in $A(\Gamma)^s$ is generated by χ_8, χ_{10}, χ_{12}, χ_{16}. If we put $S_k(\Gamma)^s = A_k(\Gamma)^s \cap S_k(\Gamma)$, the dimension formulae of symmetric Hermitian modular forms and cusp forms are given easily by these facts as follows.

Corollary 7.3.

$$\begin{aligned} \sum_{k=0}^{\infty} \dim A_k(\Gamma)^s t^k &= \frac{1+t^{16}}{(1-t^4)(1-t^8)(1-t^{10})(1-t^{12})^2}, \\ \sum_{k=0}^{\infty} \dim S_k(\Gamma)^s t^k &= \frac{t^8+t^{10}+t^{12}+t^{16}-t^{18}-t^{20}-t^{22}+t^{30}}{(1-t^4)(1-t^8)(1-t^{10})(1-t^{12})^2} \end{aligned}$$

We shall give another easy interpretation of the fundamental relation. Matsumoto [20] and Hermann [15] treated also modular forms with characters. We put

$$\Gamma(1+i) = \{g \in \Gamma; g \equiv 1_4 \pmod{(1+i)}\}.$$

We denote by v the character of Γ obtained by $v(g) = \det(g)$ for $g \in \Gamma$. For a group Γ' such that $\Gamma(1+i) \subset \Gamma' \subset \Gamma$, we denote by $A_{2k}(\Gamma', v^k)^s$ the space of holomorphic functions f which satisfy $f(\tau) = f({}^t\tau)$ and

$$f(g\tau) = f(\tau)\det(c\tau+d)^{2k}\det(g)^k \quad (g \in \Gamma(1+i)).$$

We know that $\det(g) = \pm 1$. Matsumoto [20] proved that the graded ring $\oplus_{k=0}^{\infty} A_{2k}(\Gamma(1+i), v^k)^s$ is generated by algebraically independent 5 generators, e.g. θ_{1100}^2, θ_{0011}^2, θ_{1000}^2, θ_{0000}^2, θ_{1111}^2. Also he wrote down the usual theta relations among θ_m^2. Here we follow the notation of Hermann [15] p. 118. Since there is a misprint in the expression of θ_{0001}^2 in that page, we reproduce the formulas. We put $W_1 = -\theta_{1100}^2$, $W_2 = \theta_{0011}^2$, $W_3 = \theta_{1000}^2$, $W_4 = -\theta_{0000}^2$, $W_5 = \theta_{1111}^2$, $W_6 = \theta_{0100}^2$. Then the theta relations are given by

$$\sum_{i=1}^{6} W_i = 0$$

and

$$\begin{aligned} \theta_{1001}^2 &= W_1 + W_3 + W_5, & \theta_{0001}^2 &= W_1 + W_2 + W_3, \\ \theta_{0110}^2 &= W_1 + W_5 + W_6, & \theta_{0010}^2 &= W_1 + W_2 + W_6. \end{aligned}$$

Now, Hermann gave the algebraically independent 5 generators of the graded ring of symmetric modular forms $\oplus_{k=0}^{\infty} A_{2k}(\Gamma, v^k)^s$. Following his notation, we put

$$Y_{i+1} = (4W_{1+i} + 2W_{2+i} + 6W_{3+i} + 4W_{4+i} + 2W_{5+i})$$

where $i = 1,\ldots,6$ and the indices are understood to be $\pmod 6$. Then we have $W_{1+i} = Y_{1+i} + Y_{4+i} + Y_{5+i}$ and

$$\theta_{1001}^2 = Y_1 + Y_3 + Y_5, \qquad \theta_{0001}^2 = -(Y_2 + Y_3 + Y_3),$$
$$\theta_{0110}^2 = Y_3 + Y_4 + Y_5, \qquad \theta_{0010}^2 = Y_4 + Y_5 + Y_6.$$

Obviously we get $\sum_{i=1}^{6} Y_i = 0$. For any natural number k, We put

$$s_k = \sum_{i=1}^{6} Y_i^k.$$

We have $s_1 = 0$ and $s_k \in A_{2k}(\Gamma, v^k)^s$. Hermann proved that

$$\oplus_{k=0}^{\infty} A_{2k}(\Gamma, v^k)^s = \mathbb{C}[s_2, s_3, s_4, s_5, s_6].$$

Of course s_k $(k = 2, \ldots, 6)$ are algebraically independent (cf. [15]). Since $v^k = 1$ for even k, there is some overlap between this result and Freitag-Nagaoka's result.

Following Freitag [12] p. 35, we put

$$\eta_6 = \sum_{sixty} \pm(\theta_{m_1}\theta_{m_2}\theta_{m_3})^2.$$

where the summation is over 60 syzygous triples. Since $\dim A_6(\Gamma, \det)^s = 1$, two forms s_3 and η_6 are the same up to constant. Since $\Phi(\eta_6|S_2) = G_6^1$ where G_6^1 is the normalized Eisenstein series of weight 6 of $SL_2(\mathbb{Z})$ (cf. [12]), we see easily that $9s_3/4 = \eta_6$ by calculationg $\Phi(s_3|S_2)$. Now we put $\eta_{10} = s_5/5 - s_2 \cdot s_3/6$.

Proposition 7.4. *We have the following relations.*

$$\begin{aligned} 2\eta_6^2 &= \xi_{12}, \\ 9\eta_{10}^2 &= 2^{24}(\psi_4\chi_8^2 + 12\chi_8\chi_{12} + 36\chi_{10}^2), \\ 3\eta_6\eta_{10} &= 2^{12}(\psi_4^2\chi_8 + 6\psi_4\chi_{12} + 4032\chi_8^2 - 72\chi_{16}). \end{aligned}$$

In particular the relation $(\eta_6\eta_{10})^2 = \eta_6^2\eta_{10}^2$ *is nothing but the relation in Theorem 7.2.*

Proof. Since all the above relations are the relations in $A_{4k}(\Gamma, v^{2k})^s = A_{4k}(\Gamma)^s$ for $k = 3, 5, 8$, we can show the relations, in principle, by rewriting everything by algebraically independent variables Y_i $(i = 1,\ldots, 5.)$ But since this is sometimes a nuisance, we try to give a shorter proof based on other known things. The first relation of the above proposition is nothing but what was shown in [21] pp. 543–545. By this relation, we can express χ_{12} by Y_i $(1 \le i \le 5)$

without writing all fifteen complements of syzygous quadruples in the definition of χ_{12}. It is easy to write χ_{10}^2 by Y_i. By Nagaoka's relation, the right hand side of the second relation above should be square. Since $\dim A_{10}(\Gamma, v)^s = 2$, that should be the square of some linear combination of $s_2 \cdot s_3$ and s_5. By calculating the restriction of these to the diagonal of S_2, we can find the candidate. Once we find the candidate, we can check the second relation easily by comparing expressions by Y_i by computer. This proves the second relation. Finally, it is clear from Nagaoka's relation that the last relation holds up to sign. By restricting the right hand side to S_2, we get $2^{12} \cdot 6 G_6 X_{10}$. Since $\eta_6|S_2 = G_6$, we must show that $\eta_{10}|S_2 = 2^{13}X_{10}$. The expression of X_{10} by ϑ_m^4 is known by Igusa [17] (II) p. 397. On the other hand, by definition we can write η_{10} by Y_i, and we have $\theta_m^2|S_2 = \vartheta_m^4$. So, it is an easy computer calculation to get an expression of $\eta_{10}|S_2$ by ϑ_m^4. As in Igusa [17], we use $y_0 = \vartheta_{0110}^4$, $y_1 = \vartheta_{0100}^4$, $y_2 = \vartheta_{0000}^4$, $y_3 = -\vartheta_{1000}^4 - \vartheta_{0110}^4$, $y_4 = -\vartheta_{1100}^4 - \vartheta_{0110}^4$ as a set of generators. These are not algebraically independent, but the explicit relation at weight 8 is also known by Igusa. We have

$$
\begin{aligned}
&W_1|S_2 = y_0 + y_4, && W_2|S_2 = y_2 + y_3, && W_3|S_2 = -y_0 - y_3, \\
&W_4|S_2 = -y_2, && W_5|S_2 = -y_1 - y_4, && W_6|S_2 = y_1.
\end{aligned}
$$

Using these, it is not so difficult to show that

$$\eta_{10}|S_2 = 2^{13}X_{10}.$$

Hence we get the desired result. □

8 Hermitian Modular Forms Obtained from Codes

Now our task is to determine the corresponding Hermitian modular form for some of the Type II codes. The simplest of all Type II codes is the Klemm code defined in [9]. The Klemm code K_n of length n is defined as

$$K_n = R(1, 1, \ldots, 1) + RuP_n,$$

where $P_n = \{x \in \mathbb{F}_2^n |\, \mathrm{wt}_H(x) \text{ is even}\}$. If n is divisible by 4, then K_n is a Type II code [9]. Its symmetrized biweight enumerator is

$$
\begin{aligned}
\mathrm{swe}_{K_n}(X_{\overline{a}}; \overline{a} \in \overline{R^2}) = {} & h_n(X_{\overline{00}}, X_{\overline{0u}}, X_{\overline{u0}}, X_{\overline{uu}}) + h_n(X_{\overline{01}}, X_{\overline{01}}, X_{\overline{u1}}, X_{\overline{u1}}) \\
& + h_n(X_{\overline{10}}, X_{\overline{1u}}, X_{\overline{10}}, X_{\overline{1u}}) + h_n(X_{\overline{11}}, X_{\overline{1i}}, X_{\overline{1i}}, X_{\overline{11}}),
\end{aligned}
$$

where

$$
\begin{aligned}
h_n(a, b, c, d) = {} & \frac{1}{4}((a + b + c + d)^n + (a - b + c - d)^n \\
& + (a + b - c - d)^n + (a - b - c + d)^n).
\end{aligned}
$$

The following proposition is immediate.

Proposition 8.1. *If $n \equiv 0 \pmod 4$, then* $\mathrm{swe}_{K_n}(f_{\overline{a}}; \overline{a} \in \overline{R^2}) = \psi_n$.

For Type II codes of length at most 12 other than the Klemm codes, we directly compute their symmetrized biweight enumerators by constructing all codewords. The theta constants $f_{\overline{a}}$ ($a = (a_1, a_2) \in R^2$) have the following Fourier expansion:

$$f_{\overline{a}}(\tau) = \sum_{\sigma} c(\overline{a}, \sigma) \exp 2\pi i \operatorname{Tr}(\sigma\tau),$$

where σ runs through the set

$$\{\frac{1}{4}A \mid A \in M_2(\mathbb{Z}[i]),\ A = A^* \geq 0\}.$$

and

$$c(\overline{a}, \sigma) = \left| \left\{ x \in \mathbb{Z}[i]^2 \;\middle|\; \sigma = \begin{pmatrix} |x_1 + a_1/2|^2 & (x_1 + a_1/2)\overline{(x_2 + a_2/2)} \\ \overline{(x_1 + a_1/2)}(x_2 + a_2/2) & |x_2 + a_2/2|^2 \end{pmatrix} \right\} \right|.$$

From this we can directly compute sufficiently many Fourier coefficients of $\mathrm{swe}_C(f_{\overline{a}}; \overline{a} \in \overline{R^2})$ for each Type II code of length at most 12. Comparing the Fourier coefficients, we find an expression of $\mathrm{swe}_C(f_{\overline{a}}; \overline{a} \in \overline{R^2})$ in terms of Nagaoka's generators. (Another way to do this without Fourier coefficients is as follows. As we reviewed in the previous section, Freitag or Nagaoka's generators are defined by using θ_m and they belong to the ring generated by θ_m^2 except for χ_{10}. On the other hand, the fundamental relations between θ_m^2 are completely known by Matsumoto [20] as we saw in the previous section. Since each θ_m is a linear combination of $f_{\overline{a}}$ as shown in the last section, it is rather a routine calculation to express $\mathrm{swe}_C(f_{\overline{a}}(\tau); \overline{a} \in \overline{R^2})$ by Nagaoka's generators, where we do not use any Fourier coefficients.) The results are tabulated in Table 4.

Therefore we obtain the following theorem.

Theorem 8.2. *The algebra generated by the Hermitian modular forms of the form* $\mathrm{swe}_C(f_{\overline{a}}(\tau))$, *where C runs through all Type II codes, contains the algebra*

$$\mathbb{C}[\psi_4, \psi_8, \psi_{12}, \psi_{16}, \chi_{12}].$$

Note that there is another generator for the algebra of Hermitian modular forms, namely, χ_{10}. Since there is no Type II code of length 10, χ_{10} can not be obtained as the image of the symmetrized biweight enumerator of a Type II code. Even though χ_{10} can not be obtained from Type II codes, it is possible to express χ_{10} as a polynomial in the theta constants f_a's by using Lemma 7.1. Note that from the Molien series of the group G given in Section 6, we see that there is a unique invariant of degree 10 of G.

It is not known, however, whether χ_{10}^2 belongs to the algebra generated by $\mathrm{swe}_C(f_{\overline{a}}(\tau))$.

Table 4: Hermitian modular forms of weight 8 and 12

Code	Hermitian modular form
[8,2]_$2d_4$	ψ_4^2
[8,4]_e_8a	$(8\psi_8 + 7\psi_4^2)/15$
$A_{12,i}$ $(i = 1, \ldots, 15)$	$-3/4\psi_4^3 + 9/4\psi_4\psi_8 - 1/2\psi_{12} + 9216\chi_{12}$
$A_{12,16}$	$-27/16\psi_4^3 + 51/16\psi_4\psi_8 - 1/2\psi_{12} + 9216\chi_{12}$
$B_{12,i}$	$-21/16\psi_4^3 + 49/16\psi_4\psi_8 - 3/4\psi_{12} + 16128\chi_{12}$
$C_{12,i}$ $(i \neq 20, 22, 23, 24)$	$-\psi_4^3 + 3\psi_4\psi_8 - \psi_{12} + 24576\chi_{12}$
$C_{12,i}$ $(i = 20, 22, 23, 24)$	$-11/8\psi_4^3 + 27/8\psi_4\psi_8 - \psi_{12} + 24576\chi_{12}$
$D_{12,1}, D_{12,3}, D_{12,4}$	$-21/16\psi_4^3 + 57/16\psi_4\psi_8 - 5/4\psi_{12} + 34560\chi_{12}$
$D_{12,2}$	$-3/2\psi_4^3 + 15/4\psi_4\psi_8 - 5/4\psi_{12} + 34560\chi_{12}$
$E_{12,i}$	ψ_{12}
$F_{12,i}$ $(i = 1, \ldots, 5)$	$-5/4\psi_4^3 + 15/4\psi_4\psi_8 - 3/2\psi_{12} + 46080\chi_{12}$
$F_{12,6}, F_{12,7}$	$-11/8\psi_4^3 + 31/8\psi_4\psi_8 - 3/2\psi_{12} + 46080\chi_{12}$
$F_{12,8}$	$-23/16\psi_4^3 + 63/16\psi_4\psi_8 - 3/2\psi_{12} + 46080\chi_{12}$
$G_{12,1}$	$-11/8\psi_4^3 + 33/8\psi_4\psi_8 - 7/4\psi_{12} + 59136\chi_{12}$
$EE_{12,i}$ $(i = 1, 2, 3, 4)$	ψ_4^3
$EE_{12,5}, EE_{12,6}$	$(8\psi_8 + 7\psi_4^2)\psi_4/15$
$DE_{12,i}$	$\psi_4\psi_8$

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Appendix I

Here we give generator matrices of the 82 Type II codes of length 12. We denote the codes obtained from the binary Type II code X_{24} by $X_{12,i}$ for $X = A, B, C, \ldots, G, EE$ and DE, and we denote a generator matrix of a code C by $G(C)$. To save space, we list the matrix $G(C)$ using the form $g_1, g_2, \ldots$ where g_j is the j-th row of $G(C)$.

$G(A_{12,1}) = 1010010101ui, 01011010101u, 00u00000000u, 000u000000u0, 0000u00000u0, 00000u00000u,$
$000000u000u0, 0000000u000u, 00000000u0u0, 000000000u0u$

$G(A_{12,2}) = 10000000i00u, 01000101u0u1, 001000000iu0, 000110100uiu, 0000u00000u0, 00000u00000u,$
$000000u000u0, 0000000u000u$

$G(A_{12,3}) = 1001101010ui, 010000000iu0, 001001010u1u, 000u0000000u, 0000u000000u, 00000u0000u0,$
$000000u0000u, 0000000u00u0, 00000000u00u$

$G(A_{12,4}) = 10001111u1uu, 010000001uuu, 00100000uu1u, 00010000uuu1, 0000u0000u00, 00000u000u00,$
$000000u00u00, 0000000u0u00$

$G(A_{12,5}) = 10000000uiuu, 010000000u0i, 001011110u10, 000100001u00, 0000u00000u0, 00000u0000u0,$
$000000u000u0, 0000000u00u0$

$G(A_{12,6}) = 1000111110ii, 010110000uu1, 001010000i0i, 000u00000uuu, 0000u000000u, 00000u0000u0,$
$000000u000u0, 0000000u00u0, 00000000u0u0$

$G(A_{12,7}) = 1000000iu0uu, 0100000u00i0, 001000uu1u00, 00010000ui00, 000010i0u000, 0000010u0001$

$G(A_{12,8}) = 1000011uuuui, 0100001uu011, 00100011u0u1, 0001001u10u1, 000010100i01, 00000u0uuuuu,$
$000000u0000u$

$G(A_{12,9}) = 1000000ui0uu, 010000u0uui0, 001000uiuu00, 0001000u0iuu, 000010u0uu0i, 0000011u00uu$

$G(A_{12,10}) = 10001100i0uu, 010000000iu0, 001001111ui0, 00010100100i, 0000u000u0uu, 00000u00u000,$
$000000u000u0, 0000000u00u0$

$G(A_{12,11}) = 1000000ui0uu, 01000000u00i, 001000000iu0, 00010110uu10, 00001001u000, 00000u0000u0,$
$000000u000u0$

$G(A_{12,12}) = 10000001u0uu, 0100000uu0u1, 001000000iu0, 0001011uuu1u, 0000100ui0uu, 00000u0000u0,$
$000000u000u0$

$G(A_{12,13}) = 1000011uu1uu, 0100000uuui0, 00100100ii00, 0001010i0i00, 0000100uuu01, 00000u000u00,$
$000000uuuu00$

$G(A_{12,14}) = 100000uu1uuu, 01000000uiuu, 0010000iu0uu, 000100uuuu1u, 000010i0u0uu, 000001uuuuui$

$G(A_{12,15}) = 100000uiuuuu, 0100000u00i0, 0010000u1u0u, 0001000uu10u, 0000100uuu01, 0000011u0000$

$G(A_{12,16}) = 100000010ii0, 01000001i0i0, 0010001u11i1, 0001001111u1, 000010010ii, 0000011i0010$

$G(B_{12,1}) = 10001111i0ii, 010000000111, 001000001101, 000100001110, 0000u000u0uu, 00000u00u0uu,$
$000000u0u0uu, 0000000uu0uu$

$G(B_{12,2}) = 100000u0uu01, 010000u0uui0, 001000011100, 0001001011uu, 000010110100, 000001111000$

$G(B_{12,3}) = 1000011u0111, 0100000i000u, 001000001110, 000100101010, 000010101100, 00000u00000u,$
$000000u00uuu$

$G(C_{12,1}) = 100110000uui, 010001001uiu, 001000110iuu, 000u0000000u, 0000u000000u, 00000u0000u0,$
$000000u00u00, 0000000u0u00, 00000000u0u0$

$G(C_{12,2}) = 100110100uii, 010001100i0, 001000111iiu, 000u0000000u, 0000u000000u, 00000u000uuu,$
$000000u000u0, 0000000u0u00, 00000000uu00$

$G(C_{12,3}) = 10001100u0u1, 01000000iuuu, 00100000uu1u, 00010011u1u0, 0000u000000u, 00000u00000u,$
$000000u00u00, 0000000u0u00$

$G(C_{12,4}) = 10001000100i, 01001000uui1, 00100011u10u, 00011100uu01, 0000u000000u, 00000u00u0uu,$
$000000u00u00, 0000000u0u00$

$G(C_{12,5}) = 10000001uuu0, 0100000u0iuu, 00100000uuui, 0001011uuuiu, 0000100u10uu, 00000u0000u0,$
$000000u000u0$

$G(C_{12,6}) = 10001100uu0i, 010000001uuu, 00100000u0i0, 00010011ui0u, 0000u000000u, 00000u00000u,$
$000000u00u00, 0000000u0u00$

$G(C_{12,7}) = 10000011u0u1, 010000100i01, 00100000iuuu, 00010000uu1u, 0000111uuuui, 00000u0u0u0u,$
$000000u0000u$

$G(C_{12,8}) = 10000101000i, 01000100uiui, 0010010uu011, 0001010ui0u1, 00001110u0ui, 00000u00000u,$
$000000uuuuuu$

$G(C_{12,9}) = 10001000100i, 01001001uii1, 0010001100i0, 00011101u0i1, 0000u000000u, 00000u00uu0u,$
$000000u0u0uu, 0000000u00u0$

$G(C_{12,10}) = 100000ui00uu, 010000u0u1u0, 001000uuuu1u, 000100iuuuuu, 000010uu00u1, 000001u0iuu0$

$G(C_{12,11}) = 10000001uuu0, 0100000uu10u, 00100000uuui, 0001011u001u, 0000100uiu0u, 00000u0000u0,$
$000000u000u0$

$G(C_{12,12}) = 10001100000i, 01001000uiui, 001010110uii, 000100001u00, 0000u000000u, 00000u000uuu,$
$000000u000u0, 0000000u00u0$

$G(C_{12,13}) = 10000011000i, 0100001uu0i1, 0010000uiu0u, 00010000ui00, 0000111uu001, 00000u0u00uu,$
$000000u0000u$

$G(C_{12,14}) = 100000u10u0u, 010000uuui00, 001000u000i0, 0001001uuuuu, 000010u0iu0u, 000001uuu001$

$G(C_{12,15}) = 1000011u00u1, 0100000iu0uu, 00100000ui00, 0001000uiuu0, 0000100uu01u, 00000u00000u,$
$000000u0000u$

$G(C_{12,16}) = 1000000iu0uu, 0100000u1uu0, 0010000u000i, 0001011uu0i0, 00001000ui00, 00000u0000u0,$
$000000u000u0$

$G(C_{12,17}) = 100000u0i0uu, 01000000uuiu, 001000uu01u0, 00010010uu0u, 000010u0u0u1, 000001010u00$

$G(C_{12,18}) = 1000011iuuuu, 0100000uuiu0, 0010000u000i, 0001011u0ui0, 0000111u1u00, 00000u0uu0u0,$
$000000uuu0u0$

$G(C_{12,19}) = 10000001u0uu, 010000u00i00, 0010000u000i, 000100uuu010, 000010i0uuu0, 000001uui0u0$

$G(C_{12,20}) = 1000000i1uiu, 0100000u1uii, 001000011001, 0001000i001i, 0000111u0i0u, 00000u000u00,$
$000000u00u00$

$G(C_{12,21}) = 100000uui00u, 010000u1uu00, 00100000u00i, 0001001uu0u0, 000010u000i0, 0000010u0i00$

$G(C_{12,22}) = 100001101000, 0100001u11i1, 0010010u1u10, 000101001001, 0000101i1ui1, 00000u0uuu00,$
$000000u0u0uu$

$G(C_{12,23}) = 100000110001, 010000u001uu, 0010000iiuui, 000100101001, 000010iiiuu0, 000001u00uiu$

$G(C_{12,24}) = 1000001ui0iu, 0100001uu0ii, 001000101001, 00010000i01i, 00001001uu0u, 0000010u0i00$

$G(D_{12,1}) = 10001001u111, 010001101iui, 00100001100i, 00010001iiuu, 0000u00000u0, 00000u00uu0u,$
$000000u0uu0u, 0000000u0uuu$

$G(D_{12,2}) = 100000ii1iu1, 010000u11iii, 0010001iuui0, 000100i0i0i0, 00001010uui1, 000001i001i0$

$G(D_{12,3}) = 10000001u0uu, 0100000u1u11, 00100010i0i0, 000100000i0u, 000010101uui, 000001i0uui1$

$G(D_{12,4}) = 10000101u0u1, 01000000uuiu, 0010010i1uu0, 00010110i1ui, 0000100i1001, 00000u00uu0u,$
$000000u00u00$

$G(E_{12,1}) = 11111111111i, 0u000000000u, 00u00000000u, 000u0000000u, 0000u000000u, 00000u00000u,$
$000000u0000u, 0000000u000u, 00000000u00u, 000000000u0u, 0000000000uu$

$G(E_{12,2}) = 10000000001u, 010111111uui, 00100000010u, 000u0000000u, 0000u000000u, 00000u00000u,$
$000000u0000u, 0000000u000u, 00000000u00u$

$G(E_{12,3}) = 1011111111ui, 01000000001u, 00u00000000u, 000u0000000u, 0000u000000u, 00000u00000u,$
$000000u0000u, 0000000u000u, 00000000u00u, 000000000u0u$

$G(E_{12,4}) = 1000011uuuiu, 0100000000ui, 0010000010u0, 0001000100u0, 0000100001u0, 00000u0000u0,$
$000000u000u0$

$G(E_{12,5}) = 100000uuuuiu, 010000uuuuu1, 001000uuuiuu, 000100uiuuuu, 000010iuuuuu, 000001uuiuuu$

$G(E_{12,6}) = 100000100110, 01000010010i, 001000101100, 000101100100, 000010110100, 00000u0uuuuu, 000000u00u00$

$G(E_{12,7}) = 10000000001u, 01001111uuui, 00100000100u, 00010000010u, 0000u000000u, 00000u00000u, 000000u0000u, 0000000u000u$

$G(E_{12,8}) = 100000uuuuiu, 0100000000ui, 0010001000u0, 0001000010u0, 0000100100u0, 0000010001u0,$

$G(F_{12,1}) = 1000111100u1, 01000010uuii, 001010111u01, 00011001uiu0, 0000u0000uuu, 00000u00u0uu, 000000u0000u, 0000000u0uuu$

$G(F_{12,2}) = 1000000ui0uu, 0100011uuuui, 00100001uu0u, 00010100uiu1, 0000110u0ui1, 00000u00000u, 000000u00uuu$

$G(F_{12,3}) = 1000010iuiu0, 0100010uu10i, 00100100i1uu, 0001010u0i1u, 00001110ui0u, 00000u000u00, 000000uuuuuu$

$G(F_{12,4}) = 100000u1u0u0, 0100000uuu10, 0010001u0u0u, 000100u0uu01, 0000100u10uu, 000001u00iuu$

$G(F_{12,5}) = 100000iuuuuu, 010000u00uu1, 001000uiu0u0, 000100uu00iu, 000010uuiu00, 000001u0ui0u$

$G(F_{12,6}) = 100001011ui1, 0100000uu111, 0010011ui01i, 0001000u111u, 000011000110, 00000u00u0uu, 000000uu0uu0$

$G(F_{12,7}) = 100000uuiu00, 010000100110, 001000u1u0u0, 0001000uuiii, 00001010uui1, 000001iu01ui$

$G(F_{12,8}) = 100000u01i1u, 010000uuii01, 00100011uiii, 00010001i100, 000010i11ui1, 000001iuii0u$

$G(EE_{12,1}) = 100110000i00, 010001100 0i0, 001000011001, 000u00000u00, 0000u0000u00, 00000u0000u0, 000000u000u0, 0000000u000u, 00000000u00u$

$G(EE_{12,2}) = 10000000iu00, 01000000u100, 0010110000i0, 000100110001, 0000u00000u0, 00000u0000u0, 000000u0000u, 0000000u000u$

$G(EE_{12,3}) = 100001100i00, 0100000i00u0, 0010000u0010, 00010000i00u, 00001000u00i, 00000u000u00, 000000u00u00$

$G(EE_{12,4}) = 1000000i0u00, 0100000u0100, 00100000i0u0, 00010000u010, 000010i0000u, 000001u0000i$

$G(EE_{12,5}) = 100001100i00, 0100000ii001, 0010000i0011, 0001000110i0, 00001000i011, 00000u000u00, 000000u00u00$

$G(EE_{12,6}) = 10000000ui00, 010000001u00, 0010000i0011, 000100ii0001, 0000101100i0, 000001i00011$

$G(DE_{12,1}) = 10110000000i, 010011111110, 00u00000000u, 000u0000000u, 0000u00000u0, 00000u0000u0, 000000u000u0, 0000000u00u0, 00000000u0u0, 000000000uu0$

$G(DE_{12,2}) = 100000000i0u, 010000000u0i, 001111111010, 000u000000u0, 0000u00000u0, 00000u0000u0, 000000u000u0, 0000000u00u0, 00000000u0u0$

$G(DE_{12,3}) = 10001001i000, 010001001i0, 001001000101, 000101100iuu, 0000u000u000, 00000u000u00, 000000u00uuu, 0000000uu000$

$G(DE_{12,4}) = 100001100i00, 01000001u0uu, 0010000uu0ui, 0001000u10uu, 0000100uu01u, 00000u000u00, 000000u00u00$

$G(DE_{12,5}) = 1000110000i0, 01000011u10u, 001000000u01, 000100001u00, 0000u00000u0, 00000u0000u0, 000000u00u00, 0000000u0u00$

$G(DE_{12,6}) = 10011000000i, 010001111u10, 0010000001u0, 000u0000000u, 0000u000000u, 00000u0000u0, 000000u000u0, 0000000u00u0, 00000000u0u0$

$G(DE_{12,7}) = 10000000u0i0, 01000000i0u0, 0010010i0100, 000101000101, 0000111u0i0u, 00000u000u00, 000000uu0u0u$

$G(DE_{12,8}) = 1000000i0u00, 0100000u0i00, 001000u010uu, 000100u0u0ui, 00001010u0uu, 000001u0u01u$

$G(DE_{12,9}) = 100000000i0u, 010000000u0i, 00101111u010, 0001000010u0, 0000u00000u0, 00000u0000u0, 000000u000u0, 0000000u00u0$

$G(DE_{12,10}) = 1000000i00u0, 0100000u00i0, 00100110u10u, 000100000u01, 000010001u00, 00000u000u00, 000000u00u00$

$G(DE_{12,11}) = 100001100 0i0, 0100000u0i00, 0010000ui000, 00010001uu0u, 0000100u0001, 00000u0000u0, 000000u000u0$

$$G(DE_{12,12}) = 10000000i0u0, 01000000u0i0, 0010000u0i00, 000100iu0000, 000010u10u0u, 0000010u0001$$
$$G(G_{12,1}) = 100000uiiu10, 0100001u111i, 001000u10i1u, 000100iiu01u, 000010i1i1ui, 0000010iuuii$$

Appendix II

Here we give generator matrices of the 21 Type II codes of length 16 and minimum Lee weight 8. We denote the codes obtained from the binary Type II code C by C_i and we denote a generator matrix of a code C_i by $G(C_i)$. To save space, we list the matrix $G(C_i)$ using the form $g_1, g_2, \ldots$ where g_j is the j-th row of $G(C_i)$.

$$\begin{aligned}
G(C81_1) =& 100000001iuuu11i, 01000000uuuu11i0, 00100000iui001ii, 0001000000u0i1ui,\\
& 00001000iu010i1i, 000001000111iiu0, 00000010iiiii01i, 00000001iuu0iiuu\\
G(C82_1) =& 10000100101i0111, 010001110001i1i0, 001001110111u0u1, 000100101111iiuu,\\
& 00001001110i10ii, 00000u00000u0uu0, 000000u0000uuu00, 0000000u000uu0u0,\\
& 00000000u00u00uu, 000000000u00u0uu, 0000000000u00uuu\\
G(C82_2) =& 100000010iii011i, 010000000uu1i001, 001000001i000u1u, 000100001010uu10,\\
& 0000100000ui11u0, 000001011ii10101, 000000110u0iiuuu, 0000000u000u0u0u,\\
& 00000000u00uu0u0\\
G(C82_3) =& 1000000011ii0u1, 010000011iui0i00, 001000011i1iuuii, 000100011ui01010,\\
& 000010010iu1u1iu, 000001000i0uu10i, 000000110u01100u, 0000000u0000uu0u,\\
& 00000000uu0000uu\\
G(C83_1) =& 1000001100010u0u, 0100001011iiuu11, 0010001101101i0i, 0001001111001101,\\
& 000010101011 01i1, 0000010000uui01i, 000000u0000u00uu, 0000000u000uu0u0,\\
& 00000000u0u000uu, 000000000u00uu0u\\
G(C83_2) =& 1000000011ii1011, 01000000iiuuu10u, 001000000iu0ii0u, 0001000001i0u1u0,\\
& 000010000i010iuu, 00000100i0ii111i, 00000010uiu0010i, 00000001u1uu011u\\
G(C83_3) =& 1000000010i01u0u, 01000000u001u101, 0010000001i010uu, 000100000uui00ii,\\
& 0000100011iu00u0, 0000010000u0u111, 00000010ii0uiu00, 00000001uu010110\\
G(C83_4) =& 10000000i0110iiu, 010000001i0iiu01, 00100000uui1111u, 00010000u011011i,\\
& 00001000ui01u100, 00000100i110ii1i, 0000001001110uu0, 00000001uiuiuui0\\
G(C83_5) =& 100000011010u10i, 010000011100 11i1, 00100010011iu00i, 0001000111u1uui0,\\
& 000010101100i10u, 0000011011iii010, 000000u0000uu0u0, 0000000u0000uuu0,\\
& 00000000u0uuu000, 000000000u000uuu\\
G(C83_6) =& 1000000010011u1i, 0100000001iii1u0, 0010000010101ui1, 000100001uuui1ii,\\
& 000010000ii101ui, 00000100u1i1ui1u, 000000101100iu11, 00000001iiii0iu0\\
G(C84_1) =& 100000011iu0uuu0, 0100000101iii101, 0010000101110111, 0001000011100uu0,\\
& 000010010ii0iii1, 000001010ii1i110, 0000001011uuuiu0, 0000000u0uu00u00,\\
& 00000000uu0uu0uu\\
G(C84_2) =& 100000000u11uu1u, 01000000u010iui0, 00100000110iiii1, 000100000i1uu010,\\
& 00001000i0100u1u, 000001000u100ui1, 00000010u01u0110, 00000001ii11i101\\
G(C84_3) =& 100000000i1iu01i, 0100000100u0i1i1, 001000010u0iuuui, 000100000ii1iu01,\\
& 0000100110u0i01i, 000001001i0i11ii, 0000001010i1i1ii, 0000000u0uu0u0uu,\\
& 00000000u00u0u0u\\
G(C84_4) =& 100000000i1ii0iu, 0100000001uuuu1i, 00100000u0uu1uii, 00010000uu0i00i1,\\
& 00001000i1u1i010, 000001000iuii110, 000000101iiiii01, 00000001i0iu011i
\end{aligned}$$

$G(C85_1) = 10000000001u10iu, 01000000uu010101, 0010000010uui0i0, 0001000001uuu1u1,$
$00001000iu1000iu, 00000100u101uuu1, 00000010i0iu1uuu, 00000001010101uu$

$G(C85_2) = 10000000u1111010, 010000000uiuuu1i, 00100000uui1111u, 0001000011i11i0i,$
$00001000uu0010ii, 00000100iu11i0i0, 00000010u0uiuui1, 00000001ii00uii1$

$G(C85_3) = 1000001111u1u01i, 0100000000iiiiui, 0010000011i0iii1, 0001001001u001i1,$
$00001001111u000i, 0000011101ui1000, 000000u000uu0uuu, 0000000u000u00uu,$
$00000000u0u00uu0, 000000000uu0u0u0$

$G(C85_4) = 10000000101u101i, 010000001i1i11i0, 0010000011u010ii, 00010000111ii10i,$
$00001000uu0001ii, 00000100u11u1uii, 00000010u001u011, 00000001iii00u1i$

$G(C85_5) = 100000000i0iiiui, 01000000111i1011, 00100000u100iiii, 0001000010uu0i01,$
$00001000i0ii0iiu, 000001000iiu1i0i, 000000101100u10u, 00000001iuu0iiuu$

$G(C85_6) = 10000000000uui1i, 01000000101uiuuu, 001000011iuii000, 000100000uiiii0i,$
$0000100111ii0i01, 0000010110i0u0i1, 0000001011100u11, 0000000u0u000u0u,$
$00000000u000uuu0$

$G(C85_7) = 10000000101ii0ui, 01000000010111iu, 00100000ui0101i1, 0001000010i0iu1i,$
$000010000i1iuii0, 00000100i1iu100i, 000000101iui0i10, 00000001iui01101$