Higher Weights for Ternary and Quaternary Self-Dual Codes

Steven T. Dougherty
Department of Mathematics
University of Scranton
Scranton, PA 18510
USA
T. Aaron Gulliver
Department of Electrical and Computer Engineering
University of Victoria
P.O. Box 3055, STN CSC
Victoria, BC, Canada V8W 3P6
agullive@ece.uvic.ca
and
Manabu Oura*
Division of Mathematics
Sapporo Medical University
Sapporo 060 - 8556 Japan
oura@sapmed.ac.jp

June 5, 2004

Abstract

We study higher weights applied to ternary and quaternary self-dual codes. We give lower bounds on the second higher weight and compute the second higher weights for optimal codes of length less than 24. We relate the joint weight enumerator with the higher weight enumerator and use this relationship to produce Gleason theorems. Graded rings of the higher weight enumerators are also determined.

*This work is supported in part by Northern Advancement Center for Science & Technology.

1 Introduction

In this paper we consider the theory of higher weights applied to Type III and Type IV codes. Higher weights are generalizations of Hamming weights and are also referred to as generalized Hamming weights and Wei weights ([13], [6] [14]). Type III codes are self-dual codes over $\mathbb{F}_3$, and have the property that the Hamming weight of each vector is divisible by 3. Type IV codes are self-dual codes over $\mathbb{F}_4$ with the property that the Hamming weight of each vector is even. For a complete description of self-dual codes see [11]. We shall investigate the natural weight enumerators corresponding to these weights for these codes and also give bounds on the second higher weight. The higher weights of Type I and Type II codes were studied in [4].

1.1 Notations and Definitions

Let $\mathbb{F}_q$ be a finite field with q elements. A code over $\mathbb{F}_q$ is a subset of $\mathbb{F}_q^n$. A code is linear if it is a subspace. To the space $\mathbb{F}_3^n$ we attach the standard inner product: $[v, w] = \sum v_i w_i$, and for a ternary code C we define

$$C^\perp = \{v \in \mathbb{F}_3^n \mid [v, w] = 0 \ \ \forall w \in C\}.$$

We denote the elements of $\mathbb{F}_4$ by $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$. The field $\mathbb{F}_4$ has the usual involution which fixes $0, 1$ and interchanges ω and ω^2. To the space $\mathbb{F}_4^n$ we attach the Hermitian inner-product, $v \cdot w = \sum v_i \overline{w_i}$, and for a code C define

$$C^\perp = \{v \in \mathbb{F}_4^n \mid v \cdot w = 0 \ \ \forall w \in C\}.$$

If a code C has $C \subseteq C^\perp$ then we say that C is self-orthogonal, and if $C = C^\perp$ then C is self-dual.

We shall describe the notion of higher weights following the notation in [13]. See this paper (and also [6] and [14]) for a complete description of higher weights. Let $D \subseteq \mathbb{F}_q^n$ be a linear subspace, then

$$||D|| = |Supp(D)|,$$

where

$$Supp(D) = \{i \mid \exists v \in D, \ v_i \neq 0\}.$$

For a linear code C define

$$d_r(C) = min\{||D|| \mid D \subseteq C, \ \dim(D) = r\}.$$

Notice that the minimum Hamming weight of a code C is $d_1(C)$. It also follows that $d_k = |Supp(C)|$ where k is the dimension of the code. Moreover, $d_i < d_j$ when $i < j$ (Proposition 3.1 in [13]).

The higher weight spectrum is defined as

$$A_i^r = |\{D \subseteq C \mid \dim(D) = r,\ ||D|| = i\}|.$$

This gives the following higher weight enumerators

$$W_C^{(r)} = W_C^{(r)}(y) = \sum_i A_i^r y^i,$$

or in a homogeneous form

$$W_C^{(r)} = W_C^{(r)}(x,y) = \sum_i A_i^r x^{n-i} y^i.$$

Then for each $r \leq \dim(C)$ we have a weight enumerator. If C is a code with dimension k over $\mathbb{F}_q$ then $W_C^{(r)}(1) = \begin{bmatrix} k \\ r \end{bmatrix}$, where $\begin{bmatrix} k \\ r \end{bmatrix} = \frac{(q^k-1)(q^k-q)\dots(q^k-q^{r-1})}{(q^r-1)(q^r-q)\dots(q^r-q^{r-1})}$, which is the number of subspaces of dimension r in a k dimensional space.

We adopt the notation $[s]_r = \prod_{j=0}^{r-1}(q^s - q^j)$, and for simplicity, we shall sometimes write 0 instead of $(0, \dots, 0)$.

2 Joint Weight Enumerators and Higher Weights

Since $W_{C^\perp}^{(r)}(x,y)$ involves all of $W_C^{(0)}(x,y), W_C^{(1)}(x,y), \dots, W_C^{(r)}(x,y)$, a straightforward application of invariant theory is not possible. However, we shall use the complete joint enumerator to produce a Gleason's theorem for the higher weight enumerator, in a manner similar to the technique used for binary codes in [4]. We begin with some definitions.

The g fold complete weight enumerator is defined by:

$$J_C^{(g)}(x_a) = J_C^{(g)}(x_a : a \in \mathbb{F}_q^g) = \sum_{v_1,\dots,v_g \in C} \prod_{a \in \mathbb{F}_q^g} x_a^{n_a(v_1,\dots,v_g)},$$

where $n_a(v_1, \dots, v_g)$ denotes the number of i such that $a = (v_{1_i}, \dots, v_{g_i})$.

Writing $J_C^{(g)}(x,y)$ instead of $J_C^{(g)}(x_0 = x,\ x_a = y(a \neq 0))$, two homogeneous polynomials $W_C^{(r)}(x,y)$'s and $J_C^{(g)}(x,y)$'s are related as in the following, which is proven in [4] for the binary case.

Theorem 2.1 *For C a code over $\mathbb{F}_q$*

(1) $$J_C^{(g)}(x,y) = \sum_{r=0}^{g} [g]_r W_C^{(r)}(x,y).$$

Proof. We have

$$
\begin{aligned}
W_C^{(g)}(x,y) &= \sum_{v_1,\ldots,v_g\in C} x^{n_0(v_1,\ldots,v_g)} y^{n-n_0(v_1,\ldots,v_g)} \\
&= \sum_{r=0}^{g} \sum_{\substack{v_1,\ldots,v_g\in C\\ \dim\langle v_1,\ldots,v_g\rangle = r}} x^{n_0(v_1,\ldots,v_g)} y^{n-n_0(v_1,\ldots,v_g)} \\
&= \sum_{r=0}^{g} \sum_{\substack{D\subset C\\ \dim D=r}} \sharp\{(v_1,\ldots,v_g)\in C^g; \langle v_1,\ldots,v_g\rangle = D\} \\
&\quad \times x^{n_0(v_1,\ldots,v_g)} y^{n-n_0(v_1,\ldots,v_g)}.
\end{aligned}
$$

As in [4], we can show that the number $\sharp\{(v_1,\ldots,v_g)\in C^g; \langle v_1,\ldots,v_g\rangle = D\}$ is equal to $[g]_r$. Hence we have

$$
\begin{aligned}
W_C^{(g)}(x,y) &= \sum_{r=0}^{g} \sum_{\substack{D\subset C\\ \dim D=r}} [g]_r x^{n_0(v_1,\ldots,v_g)} y^{n-n_0(v_1,\ldots,v_g)} \\
&= \sum_{r=0}^{g} [g]_r \sum_{\substack{D\subset C\\ \dim D=r}} x^{n-\|D\|} y^{\|D\|} \\
&= \sum_{r=0}^{g} [g]_r H_C^{(r)}(x,y).
\end{aligned}
$$

This completes the proof of Theorem 2.1. □

The MacWilliams type identities for higher weights, proven in [4] is a direct corollary of this theorem. The MacWilliams relations are given with a different proof in [6], [13].

Theorem 2.2 *Let C be a code over $\mathbb{F}_q$ of length n. The following identity holds:*

$$
\begin{aligned}
[g]_g W_C^{(g)}(x,y) &= J_C^{(g)}(x,y) - \sum_{\substack{F\subset \mathbb{F}^g\\ \dim F=g-1}} J_C^{(g)}\left(x_a = \left\{ \begin{array}{ll} x & \text{if } a=0 \\ y & \text{if } a\in F\setminus\{0\} \\ 0 & \text{otherwise} \end{array} \right\},\ a\in\mathbb{F}_q^g \right) \\
&\quad + \sum_{v_1,\ldots,v_g\in C} \left(\sharp\{F\subset \mathbb{F}_q^g \mid \dim F = g-1 \text{ and } P_F(v_1,\ldots,v_g)\} - 1\right) \\
&\quad \times x^{n_0(v_1,\ldots,v_g)} y^{n-n_0(v_1,\ldots,v_g)}
\end{aligned}
$$

Here $P_F(v_1,\ldots,v_g)$ means that $(v_{1i},\ldots,v_{gi})\in F$ for any i.

Proof. Theorem 2.2 follows from (1) and

$$(2)\quad \sum_{r=0}^{g-1}[g]_r W_C^{(r)}(x,y) = \sum_{\substack{F\subset\mathbb{F}_q^g\\ \dim F=g-1}} J_C^{(g)}\left(x_a=\left\{\begin{array}{ll} x & \text{if } a=0\\ y & \text{if } a\in F\backslash\{0\}\\ 0 & \text{otherwise}\end{array}\right\},\ a\in\mathbb{F}_q^g\right) - \sum_{v_1,\dots,v_g\in C}\big(\sharp\{F\subset\mathbb{F}_q^g \mid \dim F=g-1 \text{ and } P_F(v_1,\dots,v_g)\}-1\big) \times x^{n_0(v_1,\dots,v_g)}y^{n-n_0(v_1,\dots,v_g)}.$$

We shall prove (2). We have

$$\begin{aligned}\text{LHS of (2)} &= \sum_{r=0}^{g-1}\sum_{\substack{v_1,\dots,v_g\in C\\ \dim\langle v_1,\dots,v_g\rangle=r}} x^{n_0(v_1,\dots,v_g)}y^{n-n_0(v_1,\dots,v_g)}\\ &= \sum_{\substack{v_1,\dots,v_g\in C\\ 0\le\dim\langle v_1,\dots,v_g\rangle\le g-1}} x^{n_0(v_1,\dots,v_g)}y^{n-n_0(v_1,\dots,v_g)}.\end{aligned}$$

and on the other hand

$$\begin{aligned}\text{RHS of (2)} &= \sum_{\substack{F\subset\mathbb{F}_q^g\\ \dim F=g-1}}\sum_{\substack{v_1,\dots,v_g\in C\\ P_F(v_1,\dots,v_g)}} x^{n_0(v_1,\dots,v_g)}y^{n-n_0(v_1,\dots,v_g)}\\ &- \sum_{v_1,\dots,v_g\in C}\big(\sharp\{F\subset\mathbb{F}_q^g \mid \dim F=g-1 \text{ and } P_F(v_1,\dots,v_g)\}-1\big)\\ &\quad\times x^{n_0(v_1,\dots,v_g)}y^{n-n_0(v_1,\dots,v_g)}.\end{aligned}$$

Using these, we check the summation. The element $(v_1,\dots,v_g)\in C^g$ with $\dim\langle v_1,\dots,v_g\rangle\le g-1$ is counted once and only once in the summation of the RHS of (2). Hence we have the equality

$$\text{RHS of (2)} = \text{ LHS of (2)}.$$

This completes the proof of (2). □

In the case when $g=2$, the last sum of Theorem 2.2 is qx^n and we have

$$W_C^{(2)}(x,y)=\frac{1}{[2]_2}\left\{J_C^{(2)}(x,y)-\sum_{\substack{F\subset\mathbb{F}_q^2\\ \dim F=1}} J_C^{(2)}\left(x_a=\left\{\begin{array}{ll} x & \text{if } a=0\\ y & \text{if } a\in F\setminus\{0\}\\ 0 & \text{otherwise}\end{array}\right\},\ a\in\mathbb{F}_q^2\right)+qx^n\right\}.$$

Note that

$$[2]_2 = (q^2-1)(q^2-q) = \begin{cases} 6 & \text{if } q=2, \\ 48 & \text{if } q=3, \\ 180 & \text{if } q=4. \end{cases}$$

For ternary codes this gives:

Corollary 2.3 *Let C be a ternary code then*

$$\begin{aligned} W_C^{(2)}(y) &= \frac{1}{48}(J_C^{(2)}(1,y,y,y,y,y,y,y,y) - J_C^{(2)}(1,0,0,y,0,0,y,0,0) \\ &- J_C^{(2)}(1,y,y,0,0,0,0,0,0) - J_C^{(2)}(1,0,0,0,y,0,0,0,y) \\ &- J_C^{(2)}(1,0,0,0,0,y,0,y,0) + 3). \end{aligned}$$

For quaternary codes this gives:

Corollary 2.4 *Let C be a quaternary code then*

$$\begin{aligned} W_C^{(2)}(y) &= \frac{1}{180}(J_C^{(2)}(1,y,y,y,y,y,y,y,y,y,y,y,y,y,y,y) \\ &- J_C^{(2)}(1,y,y,y,0,0,0,0,0,0,0,0,0,0,0,0) \\ &- J_C^{(2)}(1,0,0,0,y,0,0,0,y,0,0,0,y,0,0,0) \\ &- J_C^{(2)}(1,0,0,0,0,y,0,0,0,0,y,0,0,0,0,y) \\ &- J_C^{(2)}(1,0,0,0,0,0,y,0,0,0,0,y,0,y,0,0) \\ &- J_C^{(2)}(1,0,0,0,0,0,0,y,0,y,0,0,0,0,y,0) + 4). \end{aligned}$$

3 Type III Codes

We notice that for ternary codes any two dimensional subspace generated by v and w consists of $\{0, v, 2v, w, w+v, w+2v, 2w, 2w+v, 2w+2v\}$. Any two linearly independent vectors of this set generate the same two dimensional subcode.

Example 1: Let C be the $[8,4,3]$ ternary code t_4^2 [11]. Then $W_C^{(0)} = 1$, $W_C^{(1)} = 8y^3+32y^6$, $W_C^{(2)} = 2y^4+16y^6+64y^7+48y^8$, $W_C^{(3)} = 8y^7+32y^8$ and $W_C^{(4)} = y^8$.

Table 3 gives d_2 and d_3 for ternary self-dual codes up to length 24. In particular, values are given for all ternary self-dual codes up to length 16, and known maximal length codes up to length 24. See [1], [9], and [10] for notation and a description of the codes.

Theorem 3.1 *Let C be a ternary self-dual code with $d_1 = 3g$, then $d_2 \geq \frac{4}{3}d = 4g$.*

Proof. Let v and w be ternary vectors and let $S = ||\langle v, w\rangle||$. Let $n_{\alpha,\beta}$ be the number of coordinates where $v_i = \alpha$ and $w_i = \beta$.

We have the following equalities:

$$S = wt(v) + n_{0,1} + n_{0,2}, \tag{3}$$
$$S = wt(w) + n_{1,0} + n_{2,0}, \tag{4}$$
$$S = wt(v + w) + n_{1,2} + n_{2,1}, \tag{5}$$
$$S = wt(2v + w) + n_{2,2} + n_{1,1}, \tag{6}$$
$$S = n_{0,1} + n_{0,2} + n_{1,0} + n_{2,0} + n_{1,2} + n_{2,1} + n_{2,2} + n_{1,1}. \tag{7}$$

Adding (3) - (6) and subtracting (7) gives

$$3S = wt(v) + wt(w) + wt(v + w) + wt(2v + w). \tag{8}$$

Since the vectors $v, w, v + w, 2v + w$ are all in the code they all have weight greater than or equal to the minimum weight $d = 3g$, and have weights a multiple of 3, so that $wt(v) = 3(g + h_1)$, $wt(w) = 3(g + h_2)$, $wt(v + w) = 3(g + h_3)$ and $wt(2v + w) = 3(g + h_4)$, where $h_i \geq 0$. Hence

$$3S = 4(3g) + 3(h_1 + h_2 + h_3 + h_4),$$

which implies

$$S = 4g + h_1 + h_2 + h_3 + h_4,$$

giving that $d_2 \geq 4g$. □

In the special case where $wt(v) = wt(w) = wt(v + w) = 3g = wt(2v + w) = d_1$ we have $d_2 = 4g$. Very often when there are a large number of minimum weight vectors this occurs (see Table 3). However, if there are few minimum weight vectors, then the bound can be exceeded. For example the code $(e_3p_{13})^+$ has $d_1 = 3$ and $d_2 = 7$.

4 Type IV Codes

A two dimensional code over $\mathbb{F}_4$ generated by v_1 and v_2 consists of $\{0, v_1, \omega v_1, \omega^2 v_1, v_2, v_2 + v_1, v_2 + \omega v_1, v_2 + \omega^2 v_1, \omega v_2, \omega v_2 + v_1, \omega v_2 + \omega v_1, \omega v_2 + \omega^2 v_1, \omega^2 v_2, \omega^2 v_2 + v_1, \omega^2 v_2 + \omega v_1, \omega^2 v_2 + \omega^2 v_1, \}$.

Example 2: Let C be the $[8, 4, 4]$ quaternary code e_8 [11]. Then $W_C^{(0)} = 1$, $W_C^{(1)} = 14y^4 + 56y^6 + 15y^8$, $W_C^{(2)} = 28y^6 + 112y^7 + 217y^8$, $W_C^{(3)} = 8y^7 + 77y^8$ and $W_C^{(4)} = y^8$.

Table 4 gives d_2 for quaternary self-dual codes up to length 24. In particular, values are given for all maximal quaternary self-dual codes up to length 18, and several known maximal codes up to length 24. See [8], [1], and [5] for notation and a description of the codes.

Theorem 4.1 *Let C be a quaternary self-dual code, then $d_2 \geq \frac{5}{4}d$.*

Proof. Let v and u be quaternary vectors and let $S = ||\langle v, u \rangle||$. Let $n_{\alpha,\beta}$ be the number of coordinates where $v_i = \alpha$ and $u_i = \beta$.

We have the following equalities:

$$S = wt(v) + n_{0,1} + n_{0,\omega} + n_{0,\overline{\omega}}, \tag{9}$$

$$S = wt(u) + n_{1,0} + n_{\omega,0} + n_{\overline{\omega},0}, \tag{10}$$

$$S = wt(v + u) + n_{1,1} + n_{\omega,\omega} + n_{\overline{\omega},\overline{\omega}}, \tag{11}$$

$$S = wt(\omega v + u) + n_{1,\omega} + n_{\omega,\overline{\omega}} + n_{\overline{\omega},1}, \tag{12}$$

$$S = wt(\overline{\omega} v + u) + n_{1,\overline{\omega}} + n_{\omega,1} + n_{\overline{\omega},\omega}, \tag{13}$$

$$S = n_{0,1} + n_{0,\omega} + n_{0,\overline{\omega}} + n_{\omega,0} + n_{\overline{\omega},0} + n_{1,1} + n_{\omega,\omega} + n_{\overline{\omega},\overline{\omega}} + n_{1,\omega} + n_{\omega,\overline{\omega}} + n_{\overline{\omega},1} n_{1,\overline{\omega}} + n_{\omega,1} + n_{\overline{\omega},\omega}. \tag{14}$$

Adding (9) - (13) and subtracting (14) gives

$$4S = wt(v) + wt(u) + wt(v + u) + wt(\omega v + u) + wt(\overline{\omega} v + u).$$

Since the vectors $v, u, v + u, \omega v + u, \overline{\omega} v + w$ are all in the code they all have weight greater than or equal to the minimum weight d, so that $wt(v) = d_1 + h_1$, $wt(u) = d_1 + h_2$, $wt(v + u) = d_1 + h_3$ $wt(\omega v + u) = d_1 + h_4$ and $wt(\overline{\omega} v + u) = d_1 + h_5$, where $h_i \geq 0$. Hence

$$4S = 5d + \sum h_i,$$

which implies

$$S = \frac{5}{4}d + \frac{5}{4}\sum h_i,$$

giving the result. □

This result is not as clean as that for ternary codes because for the bound to be met d must be divisible by 4, which is not as common and it requires more for all of the h_i to be 0.

5 Invariants and Gleason Type Theorems

Let ξ be a complex cubic root of unity and define

$$T_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{pmatrix},$$

and

$$T_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

The following MacWilliams relations are well known. Let $C_1, C_2, \ldots, C_g$ be codes in F_q^n and let $\tilde{C}$ denote either C or $C^{\perp}$. Then

$$J_{\tilde{C}_1,\tilde{C}_2,\ldots,\tilde{C}_g}(X_{\mathbf{a}}) = \frac{1}{\prod_{i=1}^{g} |C_i|^{\delta_{\tilde{C}_i}}} \cdot (\otimes_{i=1}^{g} T^{\delta_{\tilde{C}}}) J_{C_1,\ldots,C_g}(X_{\mathbf{a}})$$

where

$$\delta_{\tilde{C}} = \begin{cases} 0 & \text{if } \tilde{C} = C, \\ 1 & \text{if } \tilde{C} = C^{\perp}. \end{cases}$$

and T is T_3 if $q = 3$ and T is T_4 if $q = 4$.

Let $S_3 = \frac{1}{\sqrt{3}} T_3$, $S_4 = \frac{1}{2} T_4$ and I_ℓ denote the identity of order ℓ.

If C is a ternary self-dual code then $J^g(X)$ is held invariant by

$$M_A^3 = \otimes_{i=1}^{g} S_3^{\delta_A(i)},$$

where

$$\delta_A(i) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases},$$

and A runs over all subsets of $\{1, 2, \ldots, g\}$. Denote the set of all such matrices M_A^3 by Ω_1^3. In addition, $J^g(X)$ is held invariant by iI_{3^g} since the length of the code is divisible by 4.

Since the weight of every vector in a ternary self-dual code is divisible by 3, $J^g(X)$ is held invariant by K_B where

$$(K_B^3)_{ii} = \begin{cases} \xi & \text{if } i \in B \\ 0 & \text{if } i \notin B \end{cases},$$

and $(K_B)_{ij} = 0$ if $i \neq j$, where B runs over all subsets of $\{1, 2, \ldots, g\}$ that give the weight of a vector, that is $J(\chi_B(1), \ldots, \chi_B(3^g - 1)) = wt(v)$ for some vector in C. Denote the set of all such matrices K_B^3 by Ω_2^3.

If C is a self-dual code over $\mathbb{F}_4$ then $J^g(X)$ is held invariant by

$$M_A^4 = \otimes_{i=1}^{g} S_4^{\delta_A(i)},$$

where

$$\delta_A(i) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases},$$

and A runs over all subsets of $\{1, 2, \ldots, g\}$. Denote the set of all such matrices M_A^4 by Ω_1^4. In addition, $J^g(X)$ is held invariant by $-I_{4^g}$ since the length of the code is divisible by 2.

Since the weight of every vector in a Type IV code is divisible by 2, $J^g(X)$ is held invariant by K_B where

$$(K_B^4)_{ii} = \begin{cases} -1 & \text{if } i \in B \\ 0 & \text{if } i \notin B \end{cases},$$

and $(K_B)_{ij} = 0$ if $i \neq j$, where and B runs over all subsets of $\{1, 2, \ldots, g\}$ that give the weight of a vector, that is $J(\chi_B(1), \ldots, \chi_B(4^g - 1)) = wt(v)$ for some vector in C. Denote the set of all such matrices K_B^4 by Ω_2^4.

Define the following groups:

$$G_3^g = \langle \Omega_1^3, \Omega_2^3, iI_{3^g} \rangle$$

$$G_4^g = \langle \Omega_1^4, \Omega_2^4, iI_{4^g} \rangle$$

Let R_3^g be the ring of invariants for G_3^g and R_4^g be the ring of invariants for G_4^g. Then we have the following theorems.

Theorem 5.1 *Let C be a ternary self-dual code. Then $W_C^{(2)}(y)$ is of the form*

$$\begin{aligned}
&\tfrac{1}{48} \; (J_C^{(2)}(1, y, y, y, y, y, y, y, y) - J_C^{(2)}(1, 0, 0, y, 0, 0, y, 0, 0) \\
&- \; J_C^{(2)}(1, y, y, 0, 0, 0, 0, 0, 0) - J_C^{(2)}(1, 0, 0, 0, y, 0, 0, 0, y) - J_C^{(2)}(1, 0, 0, 0, 0, y, 0, y, 0) + 3),
\end{aligned}$$

where J is an element of R_3^2.

Proof. Follows from Theorem 2.3. □

Theorem 5.2 *Let C be a quaternary self-dual code. Then $W_C^{(2)}(y)$ is of the form*

$$\begin{aligned}
&\tfrac{1}{180} \; (J_C^{(2)}(1, y, y, y, y, y, y, y, y, y, y, y, y, y, y, y) \\
&- \; J_C^{(2)}(1, y, y, y, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \\
&- \; J_C^{(2)}(1, 0, 0, 0, y, 0, 0, 0, y, 0, 0, 0, y, 0, 0, 0) \\
&- \; J_C^{(2)}(1, 0, 0, 0, 0, y, 0, 0, 0, 0, y, 0, 0, 0, 0, y) \\
&- \; J_C^{(2)}(1, 0, 0, 0, 0, 0, y, 0, 0, 0, 0, y, 0, y, 0, 0) \\
&- \; J_C^{(2)}(1, 0, 0, 0, 0, 0, 0, y, 0, y, 0, 0, 0, 0, y, 0) + 4)
\end{aligned}$$

where J is an element of R_4^2.

Proof. Follows from Theorem 2.4. □

Let

$$\mathcal{T}_q^g = \frac{1}{\sqrt{q^t}} \begin{pmatrix} 1 & q^g - 1 \\ 1 & -1 \end{pmatrix}.$$

The fact that this matrix holds $J_C^{(g)}(x, y)$ invariant for a self-dual code follows directly by collapsing variables of the MacWilliams relation for the complete joint weight enumerator.

If C is a ternary self-dual code then $J_C^{(g)}(x, y)$ is held invariant by $\mathcal{T}_3$ and by iI_2 since the length is divisible by 4.

If C is a Type IV code then $J_C^{(g)}(x, y)$ is held invariant by $\mathcal{T}_4$ and by $-I_2$ since the length is divisible by 2.

Define the following groups:

$$\mathcal{G}_3^g = \langle T_3^g, iI_2 \rangle$$

$$\mathcal{G}_4^g = \langle T_4^g, -I_2 \rangle$$

Let $\mathcal{R}_3^g$ be the ring of invariants for $\mathcal{G}_3^g$ and $\mathcal{R}_4^g$ be the ring of invariants for $\mathcal{G}_4^g$.

Given the relationship stated in Theorem 2.1, namely that $J_C^{(g)}(x, y) = \sum_{r=0}^{g} [g]_r W_C^{(r)}(x, y)$, we have the following theorem.

Theorem 5.3 *Let C be a code over $\mathbb{F}_q$, with $q = 3$ or 4, then*

$$\sum_{r=0}^{g} [g]_r W_C^{(r)}(x, y) \in R_k^g$$

where $k = 3$ for ternary codes and $k = 4$ for quaternary codes.

6 Rings of the Higher Weight Enumerators

Let $\mathfrak{W}_{\mathbb{F}_q}^{(g)}$ be a graded ring generated by the higher weight enumerators $W_C^{(r)}(x, y)$ $(0 \leq r \leq g)$ of self-dual codes C over $\mathbb{F}_q$. In this section, we determine the structures of $\mathfrak{W}_{\mathbb{F}_q}^{(g)}$ for $q = 3$ and 4.

Theorem 6.1 *We have*

$$\mathfrak{W}_{\mathbb{F}_3}^{(0)} = \mathbb{C}[x^4],$$
$$\mathfrak{W}_{\mathbb{F}_3}^{(1)} = \mathbb{C}[x^4, y^{12}] \left(1 \oplus xy^3 \oplus (xy^3)^2 \oplus (xy^3)^3\right).$$

Proof. The first assertion follows from the fact that $W_C^{(0)}(x, y) = x^n$ for any code C and that a self-dual ternary code exists if and only if $n \equiv 0 \pmod 4$.

We shall prove the second assertion. For any ternary self-dual C, $J_C^{(1)}(x,y)$ can be written as a polynomial of x^4+8xy^3 and $x^{12}+264x^6y^6+440x^3y^9+24y^{12}$ (Gleason, see Theorem 28 in [2], p. 202). From this fact, we know that $\mathfrak{W}_{\mathbb{F}_3}^{(1)}$ is a finitely generated ring by $x^4,\ xy^3,\ y^{12}$ over $\mathbb{C}$. Using the Gröbner basis (*cf.* [3], [12]), we have

$$\begin{aligned}\mathfrak{W}_{\mathbb{F}_3}^{(1)} &= \mathbb{C}[x^4, xy^3, y^{12}] \\ &= \mathbb{C}[x^4, y^{12}]\left(1 \oplus xy^3 \oplus (xy^3)^2 \oplus (xy^3)^3\right).\end{aligned}$$

This completes the proof of Theorem 6.1. □

In order to determine the ring $\mathfrak{W}_{\mathbb{F}_3}^{(g)}$, $g \geq 2$, we need two lemmas.

Lemma 6.2 *For any $g \geq 1$ and for any ternary self-dual code C, we have*

$$A_1^g = A_2^g = A_5^g = 0.$$

Proof. Since the weight of any element of a ternary self-dual code is divisible by three, we have $A_1^g(C) = A_2^g(C) = A_5^1(C) = 0$ for any $g \geq 1$.

We shall show that $A_5^g(C) = 0$ for $g \geq 2$. For $g = 2$, the weight of two linearly independent elements is four or five and this cannot occur. For $g \geq 3$, we take three linearly independent elements which should have the following forms:

$$\begin{aligned} v_1 &= (*, *, *, 0, 0, \ldots), \\ v_2 &= (0, *, *, *, 0, \ldots), \\ v_3 &= (0, *, *, 0, *, \ldots), \end{aligned}$$

where $*$ denotes a non-zero element of $\mathbb{F}_3$. Since the weight of the sum $v_1 + v_2$ is divisible by three, we may put

$$\begin{aligned} v_1 &= (*, a, b, 0, 0, \ldots), \\ v_2 &= (0, -a, c, *, 0, \ldots), \end{aligned}$$

where $a \neq 0, b + c \neq 0$. The same argument applies to the sum $v_2 + v_3$ and $v_1 + v_3$ and we have three cases:

Case 1: $v_3 = (0, a, -b, 0, *, \ldots)$. The weight of $v_1 + v_2 + v_3 = (*, a, c, *, *, \ldots)$ is five. This cannot occur since the weight is divisible by three because of self-duality.

Case 2: $v_3 = (0, -a, -c, 0, *, \ldots)$. The weight of $v_1 + v_2 + v_3 = (*, -a, b, *, *, \ldots)$ is five. This cannot occur.

Case 3: $v_3 = (0, d, -b, 0, *, \ldots), a + d \neq 0, b = c$. The weight of $v_1 + v_2 + v_3 = (*, d, b, *, *, \ldots)$ is five. This cannot occur.

This completes the proof of Lemma 6.2. □

Lemma 6.3 *Any element of the set* $\{0, 3, 4, l \in \mathbb{Z}_{\geq 6}\}$ *can be written once and only once in the form* $4a + 3b,\ a \in \mathbb{Z}_{\geq 0},\ b \in \{0, 1, 2, 3\}$.

Proof. Trivial. □

Theorem 6.4 *For* $g \geq 2$*, we have*

$$\mathfrak{W}_{\mathbb{F}_3}^{(g)} = \mathbb{C}[x^4, y^4]\left(1 \oplus xy^3 \oplus (xy^3)^2 \oplus (xy^3)^3\right).$$

Proof. The ternary tetra code t_4 has $W_{t_4}^{(1)}(x, y) = y^4$. Hence $\mathfrak{W}_{\mathbb{F}_3}^{(1)}$ contains the ring $\mathbb{C}[x^4, y^4, xy^3]$ which can be written as

$$\mathbb{C}[x^4, y^4, xy^3] = \mathbb{C}[x^4, y^4]\left(1 \oplus xy^3 \oplus (xy^3)^2 \oplus (xy^3)^3\right). \tag{15}$$

In order to prove the equality of two rings, $\mathfrak{W}_{\mathbb{F}_3}^{g}$ and $\mathbb{C}[x^4, y^4, xy^3]$, it suffices to show that, for any ternary self-dual code C of length $n = 4c$, any term $x^{n-i}y^i$ whose coefficient A_i^g doesn't vanish is contained in the right hand side of (15). By Lemma 6.2, we have $i \neq 1,\ 2,\ 5$. In this case, there exist uniquely $a \in \mathbb{Z}_{\geq 0}$ and $b \in \{0,\ 1,\ 2,\ 3\}$ such that $i = 4a + 3b$. Hence

$$\begin{aligned} x^{n-i}y^i &= x^{4c-(4a+3b)}y^{4a+3b} \\ &= (x^4)^{c-a-b}(y^4)^a(xy^3)^b, \end{aligned}$$

and this is contained in the right hand side of (15). This completes the proof of Theorem 6.4. □

Similar results hold for the $\mathfrak{W}_{\mathbb{F}_4}^{(g)}$.

Theorem 6.5 *We have*

$$\begin{aligned} \mathfrak{W}_{\mathbb{F}_4}^{(0)} &= \mathbb{C}[x^2], \\ \mathfrak{W}_{\mathbb{F}_4}^{(1)} &= \mathbb{C}[x^2, y^2]. \end{aligned}$$

For $g \geq 2$*, we have*

$$\mathfrak{W}_{\mathbb{F}_4}^{(g)} = \mathbb{C}[x^2, y^2](1 \oplus xy^5).$$

Proof. The first assertion follows from the fact that $W_C^{(0)}(x, y) = x^n$ for any code C and that a self-dual quaternary code exists if and only if $n \equiv 0 \pmod 2$.

We shall prove the second assertion. For any quaternary self-dual C, $J_C^{(1)}(x, y)$ can be written as a polynomial of $x^2 + 3y^2$ and $x^6 + 45x^2y^4 + 18y^6$ (Theorem 30 in [2], p. 203).

From this fact, we know that $\mathfrak{W}_{\mathbb{F}_4}^{(1)}$ is a finitely generated ring in $x^2,\ y^2$ over $\mathbb{C}$ and the result follows.

Finally, we shall show the case for $g \geq 2$. The hexacode h_6 has $W_{h_6}^{(2)}(x,y) = 6xy^5 + 15y^6$. Hence $\mathfrak{W}_{\mathbb{F}_4}^{g}$ contains $\mathbb{C}[x^2, y^2, xy^5]$, which can be written as

$$\mathbb{C}[x^2, y^2, xy^5] = \mathbb{C}[x^2, y^2](1 \oplus xy^5). \tag{16}$$

In order to prove the equality of the two rings, $\mathfrak{W}_{\mathbb{F}_4}^{g}$ and $\mathbb{C}[x^2, y^2, xy^5]$, we need two claims.

Claim 1: $A_1^g = A_3^g = 0$ for any $g \geq 1$ and for any quaternary self-dual code.

Claim 2: Any element of the set $\{0, 2, l \in \mathbb{Z}_{\geq 4}\}$ can be written once and only once in the form $2a + 5b,\ a \in \mathbb{Z}_{\geq 0},\ b \in \{0, 1\}$.

These claims correspond to Lemmas 6.2, 6.3 in the $\mathbb{F}_3$ case, respectively and so we omit the proofs. Now take, for any quaternary self-dual code C of length $n = 2c$, any term $x^{n-i}y^i$ whose coefficient A_i^g doesn't vanish. By Claim 1, we have that $i \neq 1,\ 3$. In this case, there exist a unique $a \in \mathbb{Z}_{\geq 0}$ and $b \in \{0,\ 1\}$ such that $i = 2a + 5b$. Hence we have

$$\begin{aligned} x^{n-i}y^i &= x^{2c-(2a+5b)}y^{2a+5b} \\ &= (x^2)^{c-a-3b}(y^2)^a(xy^5)^b, \end{aligned}$$

and this is contained in the right hand side of (16).

This completes the proof of Theorem 6.5. □

References

[1] J.H. Conway, V. Pless and N.J.A. Sloane, "Self-dual codes over GF(3) and GF(4) of length not exceeding 16," *IEEE Trans. Inform. Theory*, vol. 25, pp. 312–322, 1979.

[2] J.H. Conway and N.J.A. Sloane, *Sphere packings, lattices and groups*, 2nd Ed., Springer, New York, 1988.

[3] D. Cox, J. Little, D. O'Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, 2nd Ed., Springer, New York, 1997.

[4] S.T. Dougherty, T.A. Gulliver and M. Oura, "Higher weights and graded rings for binary self-dual codes," *Discrete Appl. Math.*, (to appear).

[5] T.A. Gulliver, "Optimal double circulant self-dual codes over GF(4)," *IEEE Trans. Inform. Theory*, vol. 46, pp. 271–274, Jan. 2000.

[6] T. Klove, "Support weight distributions of linear codes," *Discrete Math.*, vol. 106/107, pp. 311–316, 1992.

[7] F.J. MacWilliams, C.L. Mallows and N.J.A. Sloane, "Generalizations of Gleason's theorem on weight enumerators of self-dual codes," *IEEE Trans. Inform. Theory*, vol. 18, pp. 794–805, 1972.

[8] F.J. MacWilliams, A.M. Odlyzko and N.J.A. Sloane, "Self-dual codes over GF(4)," *J. Combin. Theory Ser. A*, vol. 25, pp. 288–318, 1978.

[9] C.L. Mallows, V. Pless, and N.J.A. Sloane, "Self-dual codes over $GF(3)$," *SIAM J. Appl. Math*, vol. 31, pp. 649–666, 1976.

[10] V. Pless, N.J.A. Sloane, and H.N. Ward, "Ternary codes of minimum weight 6, and the classification of the self-dual codes of length 20," *IEEE Trans. Inform. Theory*, vol. 26, pp. 305–316, May 1980.

[11] E.M. Rains and N.J.A. Sloane, *Self-Dual Codes*, in V. S. Pless and W. C. Huffman (Eds.), *Handbook of Coding Theory*, Elsevier, Amsterdam, pp. 177–294, 1998.

[12] B. Sturmfels, *Algorithms in Invariant Theory*, Texts and Monographs in Symbolic Computation, Springer-Verlag, New York, 1993.

[13] M.A. Tsfasman and S.G. Vladut, "Geometric approach to higher weights," *IEEE Trans. Inform. Theory*, vol. 41, pp. 1564–1588, 1995.

[14] V.K. Wei, "Generalized Hamming weights for linear codes," *IEEE Trans. Inform. Theory*, vol. 37, pp. 1412-1418, 1991.

Table 1: Higher Weight Enumerators for the $[12, 6, 6]$ Golay Code

W^1	W^2	W^3	W^4	W^5	W^6	weight i
132						6
	495					8
220	880	220				9
	2970	1980	66			10
	3960	9900	1320	12		11
12	2706	21780	9625	352	1	12

Table 2: The Genus 6 Weight Enumerator for a the $[12, 6, 6]$ Golay Code

coefficient of y^i	weight i
1	0
96096	6
261621360	8
84184100000	9
18386225938080	10
2433667533897600	11
147642497091345984	12

Table 3: Ternary Self-Dual Codes with $n \leq 24$

n	Code	d_1	d_2	d_3
4	t_4	3	4	
8	t_4^2	3	4	7
12	e_3^{4+}	3	6	8
12	e_4^3	3	4	7
12	g_{12}	6	8	9
16	e_4^4	3	4	7
16	$(e_3^4 e_4)^+$	3	4	7
16	$g_{12} e_4$	3	4	9
16	$(e_3^4 f_4)^+$	3	6	8
16	$(e_3^2 g_{10})^+$	3	6	9
16	$(e_3 p_{13})^+$	3	7	9
16	f_8^{2+}	6	8	10
20	$10 f_2$	6	8	11
20	$4 f_4 + 2 f_2$	6	8	10
20	$5 f_4$	6	8	10
20	$4 f_5$	6	8	11
20	$2 g_9 + f_2$	6	8	9
20	$2 f_{10}$	6	8	9
24	XQ_{23}	9	12	14
24	$S(24)$	9	12	14

Table 4: Quaternary Self-Dual Codes with $n \leq 24$

n	Code	d_1	d_2
4	i_2^2	2	4
6	h_6	4	5
8	e_8	4	6
10	d_{10}^+	4	5
10	e_5^{2+}	4	6
12	d_{12}^+	4	6
12	$(e_7e_5)^+$	4	5
12	d_6^{2+}	4	6
12	d_4^{3+}	4	7
14	d_{14}^+	4	6
14	e_7^{2+}	4	6
14	$(d_8e_5f_1)^+$	4	5
14	$(e_5^2f_1)^+$	4	5
14	$(d_8d_6)^+$	4	6
14	$(d_6^2f_2)^+$	4	6
14	$(d_6d_4^2)^+$	4	6
14	$(d_4^3f_2)^+$	4	7
14	$(d_4^2 1_6)^+$	4	7
14	$q14$	6	8
16	f_8^{2+}	6	8
16	$(f_5^2 1_6)^+$	6	8
16	1_{16}^+	6	8
16	f_4^{4+}	6	8
18	S_{18}	8	10
20	C_{20}	8	10
22	C_{22}	8	10
24	$C_{24,1}$	8	12
24	$C_{24,2}$	8	10
24	$C_{24,3}$	8	11
24	$C_{24,4}$	8	10