ON THE INTEGRAL RING SPANNED BY GENUS TWO WEIGHT ENUMERATORS

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Abstract. It is known that the weight enumerator of a self-dual doubly-even code in genus $g = 1$ can be uniquely written as an isobaric polynomial in certain homogeneous polynomials with *integral* coefficients. We settle the case where $g = 2$ and prove the non-existence of such polynomials under some conditions.

1. Introduction. In this paper we deal with binary self-dual doubly-even codes only. We refer to [8], [3], [7] for the general facts on coding theory. We shall first recall our problem in the case where $g = 1$, which explains what this paper concerns about. It is known that the weight enumerator of any self-dual doubly-even code can be uniquely written as an isobaric polynomial in $\varphi_8 = x^8 + 14x^4y^4 + y^8$ and $\varphi_{24} = x^4y^4(x^4 - y^4)^4$ with integral coefficients ([5], [10]). We note that φ_{24} itself is not the weight enumerator of a code but a linear combination of the weight enumerators with rational coefficients.

We shall add a few words on this basis. We consider the elements in $\mathbf{Z}[x, y]$ for simplicity. The choice of φ_8 is unique (up to ± 1) since there exists a unique self-dual doubly-even code d_8^+ of length 8. Next we assume that another homogeneous polynomial ξ of degree 24 has the property in question, i.e., the weight enumerator of any self-dual doubly-even code can be written as an isobaric polynomial in φ_8 and ξ with integral coefficients. We put $\xi = ax^{24} + bx^{20}y^4 + \cdots$, $a, b \in \mathbf{Z}$, in which the unwritten part consists of terms of degree less than 20 in x. There are 85 classes self-dual doubly-even codes of length 32([1], [2]) and the weight enumerator of these classes should be written as $m\varphi_8^4 + n\varphi_8\xi$, in which m, n are integers. Examining these conditions for all classes, we know that $-42a + b$ must be a divisor of 1. We have that $\xi = a\varphi_8^3 \pm \varphi_{24}$ and conversely, such ξ has the said property.

In the rest of this paper we restrict ourselves to the case where $g = 2$ when considering the weight enumerators. Let C be a binary self-dual doubly-even code and $W_C = W_C(x, y, z, w)$ the weight enumerator of C in genus 2 (*cf.* [6], [4], [9]). We remark that W_C is symmetric in x, y, z, w. We shall denote by $\mathfrak{W}$ the graded ring over the field $\mathbf{C}$ of complex numbers generated by W_C of all self-dual doubly-even codes. The degree d-part $\mathfrak{W}_d$ of $\mathfrak{W}$ is a finite dimensional vector space over $\mathbf{C}$. Let d_{4k}^+ be a self-dual doubly-even code of length $4k$, generated by $2k$ elements

$$\begin{array}{c}(1,1,1,1,0,0,\cdots,0,0,0,0),\\ (0,0,1,1,1,1,\cdots,0,0,0,0),\\ \ddots\\ (0,0,0,0,0,0,\cdots,1,1,1,1),\\ (1,0,1,0,1,0,\cdots,1,0,1,0),\end{array}$$

and g_{24} the extended Golay code of length 24. Then the four elements $W_{d_8^+}, W_{d_{24}^+}, W_{g_{24}}, W_{d_{40}^+}$ are algebraically independent over $\mathbf{C}$ and the graded ring $\mathfrak{W}$ is a free $\mathbf{C}[W_{d_8^+}, W_{d_{24}^+}, W_{g_{24}}, W_{d_{40}^+}]$-module with a basis 1, $W_{d_{32}^+}$. The dimension formula of this ring is

$$\sum_{d\geq 0}(\dim \mathfrak{W}_d)\, t^d = \frac{1+t^{32}}{(1-t^8)(1-t^{24})^2(1-t^{40})}$$
$$= 1 + t^8 + t^{16} + 3t^{24} + 4t^{32} + 5t^{40} + 8t^{48} + 10t^{56} + \cdots .$$

We always keep this formula in mind through the next section.

2. Result. For the proof of our theorem, we shall construct homogeneous polynomials $X_8, X_{24}, Y_{24}, X_{32}, X_{40}$ of degrees $8, 24, 24, 32, 40$, respectively. This is done by analyzing the vector spaces $\mathfrak{W}_d,\ d = 8, 24, 32, 40$.

(degree 8) The extended Hamming code d_8^+ of length 8 is a unique self-dual doubly-even code of this length. We put $X_8 = W_{d_8^+}$. This polynomial is also characterized by $x^8 + \cdots$.

(degree 24) Two polynomials $X_{24},\ Y_{24}$ are characterized by

$$0x^{24} + x^{20}(y^4 + \cdots) + 0x^{18}(y^2z^2w^2) + \cdots ,$$
$$0x^{24} + 0x^{20}(y^4 + \cdots) + x^{18}(y^2z^2w^2) + \cdots ,$$

respectively. As we remarked, the weight enumerator in this paper is symmetric and $x^{20}(y^4+\cdots)$ stands for $x^{20}(y^4+z^4+w^4)$. We note that 0 as a coefficient of $x^{18}(y^2z^2w^2)$ in the first formula is not much of importance. The general form of the elements in $\mathfrak{W}_{24}$ is

$$a_0x^{24} + a_1x^{20}(y^4 + \cdots) + a_2x^{18}(y^2z^2w^2) + \cdots$$

and is uniquely written as

$$a_0X_8^3 + (-42a_0 + a_1)X_{24} + (-504a_0 + a_2)Y_{24}.$$

(degree 32) The polynomial X_{32} is characterized by

$$0x^{32} + 0x^{28}(y^4 + \cdots) + 0x^{26}y^2z^2w^2 + x^{24}(y^4z^4 + \cdots) + \cdots .$$

We remark that $0x^{32}+0x^{28}(y^4+\cdots)+\cdots$ implies that the coefficient of $x^{24}(y^8+\cdots)$ is 0. The similar remark also holds in the following (degree 40). The general form of the elements in $\mathfrak{W}_{32}$ is

$$a_0x^{32}+a_1x^{28}(y^4+\cdots)+a_2x^{26}(y^2z^2w^2)+x^{24}\left(a_3(y^8+\cdots)+a_4(y^4z^4+\cdots)\right)+\cdots$$

and is uniquely written as

$$a_0X_8^4+(-56a_0+a_1)X_8X_{24}+(-672a_0+a_2)X_8Y_{24}+(784a_0-33a_1-2a_2+a_4)X_{32},$$

where $a_3 = 620a_0 + 10a_1$.

(degree 40) The polynomial X_{40} is characterized by

$$0x^{40} + 0x^{36}(y^4 + \cdots) + 0x^{34}(y^2z^2w^2) + 0x^{32}(y^4z^4 + \cdots) + x^{28}(y^4z^4w^4) + \cdots .$$

The general form of the elements in $\mathfrak{W}_{40}$ is

$$\begin{aligned}&a_0x^{40} + a_1x^{36}(y^4 + \cdots) + a_2x^{34}(y^2z^2w^2) + x^{32}(a_3(y^8 + \cdots) + a_4(y^4z^4 + \cdots))\\ &+a_5x^{30}(y^6z^2w^2 + \cdots) + x^{28}(a_6(y^{12} + \cdots) + a_7(y^8z^4 + \cdots) + a_8(y^4z^4w^4)) + \cdots\end{aligned}$$

and is uniquely written as

$$\begin{aligned}a_0X_8^5 + (-70a_0 + a_1)X_8^2X_{24} + (-840a_0 + a_2)X_8^2Y_{24} + (1960a_0 - 61a_1 - 2a_2 + a_4)X_8X_{32}\\ +(196560a_0 - 7350a_1 - 880a_2 + 150a_4 + a_8)X_{40},\end{aligned}$$

where we have the relations $a_3 = 285a_0 + 24a_1, a_5 = 84a_1 - 8a_2 + 12a_4, a_6 = 21280a_0 + 92a_1, a_7 = 225a_1 + 32a_2 + 11a_4$.

The homogeneous polynomials we have thus obtained can be written as

$$\begin{aligned}X_8 &= W_{d_8^+},\\ X_{24} &= 5\cdot 2^{-2}3^{-1}7^{-1}W_{d_8^+}^3 - 2^{-2}11^{-1}W_{d_{24}^+} - 17\cdot 2^{-1}3^{-1}7^{-1}11^{-1}W_{g_{24}},\\ Y_{24} &= -2^{-4}3^{-1}7^{-1}W_{d_8^+}^3 + 2^{-4}3^{-1}11^{-1}W_{d_{24}^+} + 2^{-2}3^{-1}7^{-1}11^{-1}W_{g_{24}},\\ X_{32} &= 67\cdot 2^{-10}3^{-1}7^{-1}W_{d_8^+}^4 - 5\cdot 2^{-7}11^{-1}W_{d_8^+}W_{d_{24}^+} - 2^{-3}3^{-1}7^{-1}11^{-1}W_{d_8^+}W_{g_{24}} + 2^{-10}W_{d_{32}^+},\\ X_{40} &= -461\cdot 2^{-13}3^{-1}5^{-1}7^{-1}41^{-1}W_{d_8^+}^5 + 13\cdot 2^{-9}3^{-1}11^{-1}41^{-1}W_{d_8^+}^2W_{d_{24}^+}\\ &\quad + 2^{-6}3^{-1}7^{-1}11^{-1}41^{-1}W_{d_8^+}^2W_{g_{24}} - 3\cdot 2^{-13}41^{-1}W_{d_8^+}W_{d_{32}^+} + 2^{-10}3^{-1}5^{-1}41^{-1}W_{d_{40}^+}.\end{aligned}$$

We note that $X_8, X_{24}, Y_{24}, X_{32}, X_{40}$ are in $\mathbf{Z}[x, y, z, w]$ and that they generate the ring $\mathfrak{W}$.

These being prepared, we prove

THEOREM. *There exist no five homogeneous polynomials of degrees* 8, 24, 24, 32, 40 *in* $\mathfrak{W} \cap \mathbf{Z}[x, y, z, w]$ *such that the weight enumerator of any self-dual doubly-even code can be written as an isobaric polynomial in these five elements with integral coefficients.*

Proof. Suppose that there exist such homogeneous polynomials of degrees $8, 24, 24, 32, 40$ satisfying the property in the theorem. As we discussed in this section, any element in $\mathfrak{W} \cap \mathbf{Z}[x, y, z, w]$ of degree at most 40 can be uniquely

written as an isobaric polynomial in $X_8, X_{24}, Y_{24}, X_{32}, X_{40}$ with integral coefficients and the five assumed polynomials are hence integral polynomials in $X_8, \ldots, X_{40}$. Therefore $X_8, \ldots, X_{40}$ also enjoy the property in the theorem, i.e., the weight enumerator of any self-dual doubly-even code can be written as

$$\sum_{i,j,k,l,m \in \mathbf{Z}_{\geq 0}} a_{ijklm} X_8^i X_{24}^j Y_{24}^k X_{32}^l X_{40}^m,$$

in which all a_{ijklm} are integers. The weight enumerator of the code d_{56}^+ is, however, written as

$$\begin{aligned} &X_8^7 + 2^3 5 \cdot 7 X_8^4 X_{24} + 2^4 3 \cdot 5 \cdot 7 \cdot 11 X_8^4 Y_{24} + 2^8 7 \cdot 23 X_8^3 X_{32} \\ &+2^{16} 7 \cdot 139 \cdot 3^{-2} X_8^2 X_{40} + 2^8 7 X_8 X_{24}^2 + 2^{10} 3 \cdot 7 \cdot 11 X_8 X_{24} Y_{24} \\ &+2^{10} 7 \cdot 6521 \cdot 3^{-2} X_8 Y_{24}^2 + 2^{11} 5 \cdot 7 X_{24} X_{32} + 2^{12} 7 \cdot 227 \cdot 3^{-1} Y_{24} X_{32}. \end{aligned}$$

This expression is unique and we get a contradiction. This completes the proof of the theorem.

If we take a self-dual doubly-even code C of length 48, and write W_C as an isobaric polynomial in $X_8, X_{24}, Y_{24}, X_{32}, X_{40}$, then we can show that the coefficients in this expression are in $\mathbf{Z}[\frac{1}{3}]$. It was, therefore, expected to find a counter example to our assumption in the proof of the theorem at this length, but it did not work out that way.

We conclude this paper by giving two comments. One is that the author does not know a solution if we exclude the assumptions on the degrees and the number of polynomials in our theorem. Another is on the case $g = 3$. In our proof, the explicit structure of the ring $\mathfrak{W}$ is crucial. The corresponding ring in $g = 3$ seems not to be fully investigated. See [11], [9], [10].

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