A numerical study of Siegel theta series of various degrees for the 32-dimensional even unimodular extremal lattices

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Abstract

Erokhin showed that the Siegel theta series associated with the even unimodular 32-dimensional extremal lattices of degree up to 3 is unique. Later Salvati Manni showed that the difference of the Siegel theta series of degree 4 associated with the two even unimodular 32-dimensional extremal lattices is a constant multiple of the square J^2 of the Schottky modular form J, which is a Siegel cusp form of degree 4 and weight 8. In the present paper we show that the Fourier coefficients of the Siegel theta series associated with the even unimodular 32-dimensional extremal lattices of degrees 2 and 3 can be computed explicitly, and the Fourier coefficients of the Siegel theta series of degree 4 for those lattices are computed almost explicitly.

[1]

1 Introduction

In [8] Erokhin showed, among the several results, that the Siegel theta series associated with the even unimodular 32-dimensional extremal lattices of degree up to 3 is unique. Much later Salvati Manni [30] showed that the difference of the Siegel theta series of degree 4 associated with two even unimodular 32-dimensional extremal lattices is a constant multiple of the square J^2 of the Schottky modular form J, which is a Siegel cusp form of degree 4 and weight 8. We remark that Salvati Manni implies Erokhin, since Siegel's Φ operator sends Siegel theta series of higher degree to theta series of lower degree, and Φ sends cusp forms to zero.

In the present paper we first show that (possibly all) the Fourier coefficients of the Siegel theta series associated with any even unimodular 32-dimensional extremal lattice of degrees 2 and 3 can be computed explicitly, and next we will show that the Siegel theta series of degree 4 associated with an even unimodular 32-dimensional extremal lattices is a linear combination of two explicitly computable Siegel modular forms of degree 4 and weight 16. One is J^2 and another is a pan theta series (we name it) $P\Theta_4(Z)$, which is obtained from the Siegel theta series associated with any one of even unimodular 32-dimensional extremal lattices and J^2. $P\Theta_4(Z)$ is showed to be independent of the choice of such a lattice $\mathcal{L}_{32}$.

[1]**2000 Mathematics Subject Classification**:Primary 11F46, Secondary11E20,11T71
keywords: theta series with spherical functions, Siegel theta series, 32-dimensional extremal lattices
Running head:Siegel theta series for the 32-dimensional extremal lattices

Our present report is intermediate in the degree 4 case. The reason for this is that we have not determined the Fourier coefficients of $\Theta_4(Z, \mathcal{L}_{32})$ explicitly. However we could show that the Fourier coefficients of $\Theta_4(Z, \mathcal{L}_{32})$ can be made explicit in the sense that if one specified Fourier coefficient of $\Theta_4(Z, \mathcal{L}_{32})$ is obtained then all the Fourier coefficients are in principle computable. One may note that the results in the present paper are free from constructions for the extremal lattices, while the number of non-isometric 32-dimensional even unimodular extremal lattices is at least one billion (King [15], Cor. 17).

Our basic tool in the present paper is to use the relations among the inner products of the vectors in the even unimodular 32-dimensional extremal lattice. Those relations were founded by Schöneberg [34] and first used by Hecke [10] and later B. Venkov (e.g. [39], [40], [41]) extensively used them to derive many interesting results. We remark that the central idea of the present paper is closely connected with that of [25]. However the starting period of the present work is much earlier than that of [25]. The delay of the completion of the present work compared to that of [25] is caused by the paucity of the conditions in 32-dimensional even unimodular extremal lattices, which forced us to appeal to the computations in coding theory.

In the coming article [21] we will compute one particular Fourier coefficient which covers the defect of the present paper. Thus our present result has made a little progress for the problem posed by Salvati Manni [30].

Acknowledgment: The authors thank Professor R. Salvati Manni for helpful comments. The authors also thank the referee of the present article for giving them some critical comments which spur them to improve some parts of the article largely.

2 Some Basic Definitions

2.1 Some definitions from lattice theory

Let $\mathbb{Z}$ be the ring of rational integers and $\mathbb{R}$ be the field of real numbers. A finitely generated $\mathbb{Z}$-module L in $\mathbb{R}^g$ with a positive definite metric is called a positive definite quadratic lattice. Since we treat only the positive definite quadratic lattices, we shall omit the adjectives "positive definite quadratic". A lattice L is integral if L satisfies $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}$ for any $\mathbf{x}, \mathbf{y} \in L$ where $(\ , \ )$ is the bilinear form associated to the metric. Two integral lattices L_1 and L_2 are said to be isometric if and only if there exists a bijective linear mapping from L_1 to L_2 preserving the metric. The maximal number of linearly independent vectors over $\mathbb{R}$ in L is called the rank of L. The dual lattice $L^{\#}$ of L is defined by

$$L^{\#} = \{\mathbf{y} \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid (\mathbf{x}, \mathbf{y}) \in \mathbb{Z}, \forall \mathbf{x} \in L\}.$$

Here $\mathbb{Q}$ is the field of rational numbers. A lattice L is even if any element $\mathbf{x}$ of L has an even norm $(\mathbf{x}, \mathbf{x})$. In an even lattice L, we say that $\mathbf{x}$ is a $2m$-vector if $(\mathbf{x}, \mathbf{x}) = 2m$ holds for some natural number m. Let $\Lambda_{2m}(L)$ be the set defined by

$$\Lambda_{2m}(L) = \{\mathbf{x} \in L \mid (\mathbf{x}, \mathbf{x}) = 2m\}. \tag{2.1}$$

A lattice L is called unimodular if $L = L^{\#}$. Even unimodular lattices exist only when $n \equiv 0 \pmod 8$. The minimal norm of a lattice is $\mathrm{Min}(L) = \min_{\mathbf{x} \in L \setminus \{0\}} (\mathbf{x}, \mathbf{x})$. When L is even unimodular of rank n it holds that (conf. [17])

$$\mathrm{Min}(L) \le 2\left[\frac{n}{24}\right] + 2.$$

Such a lattice which attains the above maximum is said to be extremal.

In the present paper we are particularly interested in any even unimodular extremal 32-dimensional lattice. We denote any one of such lattices as $\mathcal{L}_{32}$. We have $\mathrm{Min}(\mathcal{L}_{32}) = 4$.

2.2 Theta series of one complex variable associated with the even unimodular lattice

Let L be an even unimodular lattice of rank $8k$, then the (ordinary) theta series for L is defined by

$$\vartheta(z, L) = \sum_{\mathbf{x}\in L} \exp(\pi i(\mathbf{x}, \mathbf{x})z), \tag{2.2}$$

where z is a complex variable with positive imaginary part. This series is rewritten as

$$\vartheta(z, L) = \sum_{m=0}^{\infty} a(2m, L) \exp(2\pi i m z), \tag{2.3}$$

where $a(2m, L) = |\Lambda_{2m}(L)|$.

When we consider the lattice $\mathcal{L}_{32}$, then the Fourier expansion of $\vartheta(z, \mathcal{L}_{32})$ is given by

$$\vartheta(z, \mathcal{L}_{32}) = 1 + 146880\mathbf{e}(2z) + 64757760\mathbf{e}(3z) + 4844836800\mathbf{e}(4z) + \cdots ,$$

where $\mathbf{e}(z)$ is the abbreviation of $\exp(2\pi i z)$.

With the theory of modular forms Venkov [40] obtained

Proposition 2.1. *Let $\mathcal{L}_{32}$ be an even unimodular extremal 32-dimensional lattice, $\Lambda_4 = \Lambda_4(\mathcal{L}_{32})$ and $\boldsymbol{\alpha} \in \mathcal{L}_{32} \bigotimes \mathbb{R}$, then we have*

$$\sum_{\mathbf{x}\in\Lambda_4} (\mathbf{x}, \boldsymbol{\alpha})^2 = 18360 \cdot (\boldsymbol{\alpha}, \boldsymbol{\alpha}), \tag{2.4}$$

$$\sum_{\mathbf{x}\in\Lambda_4} (\mathbf{x}, \boldsymbol{\alpha})^4 = 6480 \cdot (\boldsymbol{\alpha}, \boldsymbol{\alpha})^2, \tag{2.5}$$

$$\sum_{\mathbf{x}\in\Lambda_4} (\mathbf{x}, \boldsymbol{\alpha})^6 = 3600 \cdot (\boldsymbol{\alpha}, \boldsymbol{\alpha})^3, \tag{2.6}$$

$$\sum_{\mathbf{x}\in\Lambda_4} (\mathbf{x}, \boldsymbol{\alpha})^{10} - \frac{15 \cdot (\boldsymbol{\alpha}, \boldsymbol{\alpha})}{4} \sum_{\mathbf{x}\in\Lambda_4} (\mathbf{x}, \boldsymbol{\alpha})^8 = -7560 \cdot (\boldsymbol{\alpha}, \boldsymbol{\alpha})^5. \tag{2.7}$$

In the same sort of effort we can prove

Proposition 2.2. *Let $\mathcal{L}_{32}$ be an even unimodular extremal 32-dimensional lattice and $\Lambda_6 = \Lambda_6(\mathcal{L}_{32})$, then we have*

$$\begin{aligned}
\sum_{\mathbf{x}\in\Lambda_6}(\mathbf{x},\boldsymbol{\alpha})^2 &= 12142080\cdot(\boldsymbol{\alpha},\boldsymbol{\alpha}),\\
\sum_{\mathbf{x}\in\Lambda_6}(\mathbf{x},\boldsymbol{\alpha})^4 &= 6428160\cdot(\boldsymbol{\alpha},\boldsymbol{\alpha})^2,\\
\sum_{\mathbf{x}\in\Lambda_6}(\mathbf{x},\boldsymbol{\alpha})^6 &= 5356800\cdot(\boldsymbol{\alpha},\boldsymbol{\alpha})^3,\\
\sum_{\mathbf{x}\in\Lambda_6}(\mathbf{x},\boldsymbol{\alpha})^{10}-\frac{45\cdot(\boldsymbol{\alpha},\boldsymbol{\alpha})}{8}\sum_{\mathbf{x}\in\Lambda_6}(\mathbf{x},\boldsymbol{\alpha})^8 &= -25310880\cdot(\boldsymbol{\alpha},\boldsymbol{\alpha})^5.
\end{aligned}$$

2.3 Siegel theta series

The Siegel theta series of degree g ($g \geq 2$) attached to the lattice L is defined by

$$\Theta_g(Z,L) = \sum_{\mathbf{x}_1,\ldots,\mathbf{x}_g\in L} \exp(\pi i\sigma([\mathbf{x}_1,\ldots,\mathbf{x}_g]Z)),$$

where Z is the variable on the Siegel upper-half space of degree g, $[\mathbf{x}_1,\ldots,\mathbf{x}_g]$ is a g by g square matrix whose (i,j) entry is $(\mathbf{x}_i,\mathbf{x}_j)$ and σ is the trace of the matrix.

The Siegel theta series of degree g can be expanded to

$$\Theta_g(Z,L) = \sum_{T\in\hat{\mathcal{P}}_g^s(\mathbb{Z})} a(T,L)e^{2\pi i\sigma(TZ)}.$$

Here $\hat{\mathcal{P}}_g^s(\mathbb{Z})$ is the set of positive semi-definite semi-integral symmetric square matrices of degree g, and $a(T,L) = |\{\langle\mathbf{x}_1,\ldots,\mathbf{x}_g\rangle \in L^g \mid [\mathbf{x}_1,\ldots,\mathbf{x}_g] = 2T\}|$.

Fact: A Siegel theta series of degree g associated with an even unimodular lattice L of rank $2k$ ($2k$ is a multiple of 8) is a modular form of degree g and weight k.

2.4 Fourier Coefficients of Siegel Theta series of degree 2 associated with $\mathcal{L}_{32}$

We compute some Fourier coefficients of Siegel theta series of degree 2 for $\mathcal{L}_{32}$. We give a simple lemma.

Lemma 2.3. *For any two vectors $\mathbf{x},\mathbf{y} \in \Lambda_4(\mathcal{L}_{32})$ it holds that*

$$(\mathbf{x},\mathbf{y}) = \begin{cases} \pm 4, \\ \pm 2, \\ \pm 1, \\ 0. \end{cases}$$

Proof. By the Schwartzian inequality:

$$(\mathbf{x}, \mathbf{y})^2 \leq (\mathbf{x}, \mathbf{x}) \cdot (\mathbf{y}, \mathbf{y}) = 16,$$

we have

$$|(\mathbf{x}, \mathbf{y})| \leq 4.$$

Suppose $(\mathbf{x}, \mathbf{y}) = \pm 3$, then we have $(\mathbf{x} \mp \mathbf{y}, \mathbf{x} \mp \mathbf{y}) = 2$. This is impossible because $\mathcal{L}_{32}$ does not contain any vector of norm 2. □

In the equations (2.4)∼(2.7) we take any one vector $\boldsymbol{\alpha} \in \Lambda_4(\mathcal{L}_{32})$, and we put

$$\lambda_a = |\{\mathbf{x} \in \Lambda_4 | \ (\mathbf{x}, \boldsymbol{\alpha}) = a\}|.$$

By Lemma 2.3 a must be one of $\pm 4, \pm 2, \pm 1, 0$. Note that $\lambda_4 = \lambda_{-4} = 1$ and $\lambda_a = \lambda_{-a}$ holds for $a = 2, 1$. By putting these quantities into the equation (2.4) we get

$$2 \cdot 4\lambda_2 + 2 \cdot \lambda_1 + 2 \cdot 4^2 = 18360 \cdot 4. \tag{2.8}$$

In the same way from (2.5) we get

$$2 \cdot 2^4\lambda_2 + 2 \cdot \lambda_1 + 2 \cdot 4^4 = 6480 \cdot 4^2. \tag{2.9}$$

We can solve the equations (2.8) and (2.9), and the solution is $\lambda_2 = 1240, \lambda_1 = 31744$. By the setting of λ_a's $2\lambda_4 + 2\lambda_2 + 2\lambda_1 + \lambda_0$ counts all members of the set Λ_4 non-overlappingly, hence we have

$$2\lambda_4 + 2\lambda_2 + 2\lambda_1 + \lambda_0 = |\Lambda_4| = 146880,$$

and $\lambda_0 = 80910$. The remaining equations (2.6) and (2.7) are redundant for solving λ_2 and λ_1. As a summary we have

Proposition 2.4. *For any one of elements* $\boldsymbol{\alpha} \in \Lambda_4(\mathcal{L}_{32})$ *there are* (i) 80910 *elements* $\mathbf{x} \in \Lambda_4(\mathcal{L}_{32})$ *with* $(\mathbf{x}, \boldsymbol{\alpha}) = 0$, (ii) 31744 *elements* $\mathbf{x} \in \Lambda_4(\mathcal{L}_{32})$ *with* $(\mathbf{x}, \boldsymbol{\alpha}) = 1$, (iii) 1240 *elements* $\mathbf{x} \in \Lambda_4(\mathcal{L}_{32})$ *with* $(\mathbf{x}, \boldsymbol{\alpha}) = 2$. *Consequently we have*

$$a(\mathfrak{T}_{20}, \mathcal{L}_{32}) = 146880 \cdot 80910, a(\mathfrak{T}_{21}, \mathcal{L}_{32}) = 146880 \cdot 31744, a(\mathfrak{T}_{22}, \mathcal{L}_{32}) = 146880 \cdot 1240,$$

where

$$\mathfrak{T}_{20} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \mathfrak{T}_{21} = \begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix}, \mathfrak{T}_{22} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{2.10}$$

We express binary symmetric matrix $T = \begin{pmatrix} t_{11} & t_{12}/2 \\ t_{12}/2 & t_{22} \end{pmatrix}$ as (t_{11}, t_{22}, t_{12}) to save the space. d_T is the determinant of the binary matrix $2T$.

Table 3. Fourier coefficients of Siegel-theta series of degree 2 for the lattice $\mathcal{L}_{32}$

d_T	T	$a(T, \mathcal{L}_{32})$
*12	(2,2,2)	182131200
15	(2,2,1)	4662558720
*16	(2,2,0)	11884060800

In the above table and the tables afterwards concerning the Fourier coefficients the discriminants d_T marked by $*$ indicate that the quaternary quadratic forms have imprimitive coefficients.

Remark 1. *There are similar formulas to those given in Propositions 2.1,2.2 using Λ_k ($k = 8, 10, \cdots$) instead of using Λ_4 or Λ_6, and these formulas will serve for computing the Fourier coefficients of Siegel theta series of degrees up to 3 for extremal even unimodular 32-dimensional lattices. However when the index T is reduced and has the mixed diagonal entries such as 2, 2, 3, then the computations of the Fourier coefficients would be more complicated.*

3 Siegel theta series of degree 3

3.1 Positive Definite Quadratic Forms

We do not intend to discuss the full definitions of quadratic forms, but we need some definitions of quadratic forms for the later parts of the present paper. From here we will frequently need to discuss one arithmetical property of the positive definite integral quadratic forms. Let T be a symmetric square matrix of size g written by

$$T = \begin{pmatrix} t_{11} & t_{12}/2 & \cdots & t_{1g}/2 \\ t_{12}/2 & t_{22} & \cdots & t_{2g}/2 \\ \vdots & \vdots & \vdots & \vdots \\ t_{1g}/2 & t_{2g}/2 & \cdots & t_{gg} \end{pmatrix},$$

then associated with it a quadratic form $Q_T[\xi]$ is defined by

$$Q_T[\xi] = Q_T[\xi_1, \cdots, \xi_g] = \sum_{1\le i\le j\le g} t_{ij}\xi_i\xi_j,$$

where $\xi_1, \cdots, \xi_g$ are real independent variables. Let $GL_g(\mathbb{Z})$ be the group of all unimodular square matrices of size g. Two quadratic forms $Q_{T_1}[\xi] = \sum_{1\le i\le j\le g} t^{(1)}_{ij}\xi_i\xi_j$ and $Q_{T_2}[\xi] = \sum_{1\le i\le j\le g} t^{(2)}_{ij}\xi_i\xi_j$ are said to be integrally equivalent if there is an element $U \in GL_g(\mathbb{Z})$ such that the equality $U^tT_1U = T_2$ holds, where U^t is the transposed of the matrix U. A quadratic form $Q_T[\xi]$ with real entries t_{ij} is said to be positive semi-definite if it satisfies the condition that $Q_T[\xi] \ge 0$ for any set of real numbers $\xi_1, \cdots, \xi_g$, and the form $Q_T[\xi]$ is called positive definite if it satisfies the condition: $Q_T[\xi] > 0$ for any set of real numbers $\xi_1, \cdots, \xi_g$, where at least one of them is not zero. When $Q_T[\xi]$ is positive definite the matrix T is also called positive definite. Let $\mathcal{P}_g(\mathbb{R})$ be the set of positive definite symmetric matrices whose entries are real numbers.

The Minkowski reduction theory of positive definite symmetric real matrices treats the conditions by which one can find all representatives of matrices of equivalence classes. For the precise conditions on the reduction one may consult the references [18], [37] or [42].

An element $T \in \mathcal{P}_g(\mathbb{R})$ is called semi-integral if the diagonal entries of T are all integers and the off-diagonal entries are integers or half-integers. If T is a semi-integral then the associated quadratic form $Q_T[\xi] = \sum_{1\le i\le j\le g} t_{ij}\xi_i\xi_j$ has integer coefficients, and $Q_T[\xi]$ takes integer value whenever $\xi_1, \cdots, \xi_g$ are all integers. When T is a semi-integral and positive definite the discriminant d_T of $Q_T[\xi]$ is defined by

$$d_T = \begin{cases} \det(2T) & \text{if } g \text{ is even,} \\ \frac{1}{2}\det(2T) & \text{if } g \text{ is odd.} \end{cases}$$

We denote by $\mathcal{P}_g^s(\mathbb{Z})$ the subset of $\mathcal{P}_g(\mathbb{R})$ consisting of semi-integral matrices. We will use the table of the reduced elements in $\mathcal{P}_g^s(\mathbb{Z})$ with $g = 3, 4, 5$. Thanks to N.J.A. Sloane's home page we can utilize the tables of the reduced quadratic forms of sizes 3 and 4 [1], [19]. One may note that in these tables mostly primitive forms are recorded. Here the quadratic form is called primitive if the coefficients $t_{ii}, 2t_{ij} (i \neq j)$ are coprime integers, and the forms without this condition are called imprimitive.

We add one unusual terminology. A positive definite semi-integral quadratic form (or matrix) $Q_T[\xi] = \sum_{1 \leq i \leq j \leq g} t_{ij}\xi_i\xi_j$ is called 2-special if $Q_T[\xi]$ satisfies the following property: when T is reduced then the diagonal entries of the obtained form are all 2. We give instances of 2-special quadratic forms. Viewing the table [19] we find that

$$2(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2) + \xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3 + 2\xi_1\xi_4 + 2\xi_2\xi_4 - \xi_3\xi_4$$

is a reduced primitive form of discriminant 81. This is verified to be the 2-special primitive form with the least discriminant. However there are only two imprimitive 2-special forms with smaller discriminants. The form

$$2(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2) + 2\xi_1\xi_4 + 2\xi_2\xi_4 + 2\xi_3\xi_4$$

is a 2-special imprimitive form of discriminant 64, and the form

$$2(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2) + 2\xi_1\xi_2 + 2\xi_1\xi_4 + 2\xi_3\xi_4$$

is a 2-special imprimitive form of discriminant 80. These forms are obtained from primitive forms of discriminant 4 (resp. 5) by multiplying each coefficient by 2.

There are two general questions, namely, when we are given a positive definite quadratic form with integer coefficients (i) how we know the minimal value represented by the quadratic form, and (ii) how we know that the form is 2-special. One nice (but not complete) tool for these questions is the theta series of one variable. Suppose T is in $\mathcal{P}_g^s(\mathbb{Z})$, then the theta series for $Q_T[\xi]$ is defined by

$$\vartheta(z, Q_T) = \sum_{\xi_1, \cdots, \xi_g \in \mathbb{Z}} \exp(\pi i z (\sum_{1 \leq i \leq j \leq g} t_{ij}\xi_i\xi_j)),$$

with the variable $z \in \mathfrak{H}_1$. To answer the first question above one may compute the theta series of the form in question and see the beginning terms of the expansion. To answer the second question one computes the discriminant of the form and then compute the theta series of the form. After these we compute the theta series of the reduced forms of the same discriminant with the form in question, and compare the theta series of them. The following results, which are due to A. Schiemann, are suitable for our purpose.

Proposition 3.1 (Schiemann,[33]). *Two ternary positive definite forms with real coefficients are integrally equivalent if and only if their theta series in one variable are equal.*

Proposition 3.2 (Schiemann,[32]). *Except one pair of forms two positive definite quaternary even integral quadratic forms with the same discriminant* ≤ 3000 *are integrally equivalent if and only if their theta series of one variable equal.*
The only exceptional pair of forms is given at the discriminant 1729.

In the study of Siegel theta series the knowledge about the reduction theory of positive definite semi-integral quadratic forms is indispensable. In this section we briefly described the basic definitions of quadratic forms and the references. However one may remark that in the tables [1], [19] quadratic forms (matrices T) with imprimitive coefficients are mostly omitted, but the Fourier coefficients with such T's carry important informations. In our small tables we include the values of imprimitive T's.

3.2 Some General Notations

Before going to specific cases we introduce some general notations. Let $\mathfrak{T}$ be a positive semi-definite semi-integral matrix of size s. Let $a_1, \cdots, a_s$ be s integers, and we denote by $(\mathfrak{T}, \{a_1/2, \cdots, a_s/2\}, 2)$ the matrix of size $s+1$ defined by

$$\begin{pmatrix} & & & a_1/2 \\ & \mathfrak{T} & & \vdots \\ & & & a_s/2 \\ a_1/2 & \cdots & a_s/2 & 2 \end{pmatrix}.$$

For instance $(\mathfrak{T}, \{a/2, b/2\}, 2)$ denotes the matrix

$$\begin{pmatrix} t_{11} & t_{12}/2 & a/2 \\ t_{12}/2 & t_{22} & b/2 \\ a/2 & b/2 & 2 \end{pmatrix},$$

where

$$\mathfrak{T} = \begin{pmatrix} t_{11} & t_{12}/2 \\ t_{12}/2 & t_{22} \end{pmatrix}.$$

For an s-tuple $\mathbf{x}_1, \cdots, \mathbf{x}_s \in \Lambda_4$ satisfying $[\mathbf{x}_1, \cdots, \mathbf{x}_s] = 2\mathfrak{T}$ we will use a subset of Λ_4 defined by

$$\Lambda_{a_1,\cdots,a_s}(2\mathfrak{T}; \mathbf{x}_1, \cdots, \mathbf{x}_s) = \{\mathbf{z} \in \Lambda_4 | (\mathbf{x}_i, \mathbf{z}) = a_i, i = 1, \cdots, s\},$$

and

$$\lambda_{a_1,\cdots,a_s}(2\mathfrak{T}; \mathbf{x}_1, \cdots, \mathbf{x}_s) = |\Lambda_{a_1,\cdots,a_s}(2\mathfrak{T}; \mathbf{x}_1, \cdots, \mathbf{x}_s)|.$$

With these notations we can describe the Fourier coefficient at $(\mathfrak{T}, \{a_1/2, \cdots, a_s/2\}, 2)$ by

$$a((\mathfrak{T}, \{a_1/2, \cdots, a_s/2\}, 2), \mathcal{L}) = \sum_{\substack{\mathbf{x}_1,\cdots,\mathbf{x}_s \in \Lambda_4 \\ [\mathbf{x}_1,\cdots,\mathbf{x}_s] = 2\mathfrak{T}}} \lambda_{a_1,\cdots,a_s}(2\mathfrak{T}; \mathbf{x}_1, \cdots, \mathbf{x}_s). \tag{3.1}$$

We are going to calculate the Fourier coefficients $a(T, \mathcal{L})$ of Siegel theta series of various degrees and some T's. Before doing this we recall a property of $a(T, \mathcal{L})$ and an important assumption which enables us to compute $a(T, \mathcal{L})$.

Proposition 3.3. *The Fourier coefficient $a(T, \mathcal{L})$ has the invariance property:*

$$a(T, \mathcal{L}) = a(U^t T U, \mathcal{L}), \tag{3.2}$$

where U is any unimodular matrix of the same size with that of T. (Conf.[36] formula (48)).

Modulo Theta Assumption(**MTA**): From here we may assume that the equality

$$\lambda_{a_1,\cdots,a_s}(2\mathfrak{T};\mathbf{x}_1,\cdots,\mathbf{x}_s) = \lambda_{b_1,\cdots,b_s}(2\mathfrak{T};\mathbf{y}_1,\cdots,\mathbf{y}_s)$$

holds. Here $[\mathbf{x}_1,\cdots,\mathbf{x}_s] = [\mathbf{y}_1,\cdots,\mathbf{y}_s] = 2\mathfrak{T}$ and two matrices, $(\mathfrak{T},\{a_1/2,\cdots,a_s/2\},2)$ and $(\mathfrak{T},\{b_1/2,\cdots,b_s/2\},2)$, are equivalent and both 2-special.

Remark 2. *It may occur that* $\lambda_{a_1,\cdots,a_s}(2\mathfrak{T};\mathbf{x}_1,\cdots,\mathbf{x}_s) \neq \lambda_{b_1,\cdots,b_s}(2\mathfrak{T};\mathbf{y}_1,\cdots,\mathbf{y}_s)$ *for some special cases. However in calculating Fourier coefficients* $a(T,\mathcal{L})$ *one may assume the Assumption because of* (3.2)*.*

3.3 Ternary Matrices Arising from $\mathfrak{T}_{22}$

We use the matrices (2.10) again.

First we seek all possible pairs of integers a, b under the conditions:
(i) $(\mathfrak{T}_{22},\{a/2,b/2\},2)$ is positive semi-definite,
(ii) when $(\mathfrak{T}_{22},\{a/2,b/2\},2)$ is reduced under unimodular transformations: $U^t(\mathfrak{T}_{22},\{a/2,b/2\},2)U$, the minimal value of the non-zero diagonal entries of the resulting matrix is 2.

Remark 3. *When* $(\mathfrak{T}_{22},\{a/2,b/2\},2)$ *is positive definite and satisfies the condition* (ii)*, then the matrix is* 2*-special in the sense that is introduced in Section* 3.1*. If we confine ourselves to the case when* $(\mathfrak{T}_{22},\{a/2,b/2\},2)$ *are positive definite only, then we can not use the equations* (2.4)$\sim$(2.7) *in the later arguments, since the equations include the terms that imply* $(\mathfrak{T}_{22},\{a/2,b/2\},2)$ *are not positive definite.*

Besides this there are the cases when $(\mathfrak{T}_{22},\{a/2,b/2\},2)$ *satisfies the condition* (i) *above but does not satisfy* (ii)*. It is legitimate to eliminate such cases by proving two lemmas below.*

Lemma 3.4. *Let* $\mathfrak{T}$ *be a positive definite symmetric semi-integral* 2*-special matrix of size* s (≥ 2)*. Let* $(\mathfrak{T},\{a_1/2,\cdots,a_s/2\},2)$ *be an enlarged matrix made from* $\mathfrak{T}$ *by adding half the integers* $a_1/2,\cdots,a_s/2$*. If* $(\mathfrak{T},\{a_1/2,\cdots,a_s/2\},2)$ *is positive definite and not* 2*-special, then there do not exist* $\mathbf{x}_1,\cdots,\mathbf{x}_{s+1} \in \mathcal{L}$ *so that the equation*

$$[\mathbf{x}_1,\cdots,\mathbf{x}_{s+1}] = 2(\mathfrak{T},\{a_1/2,\cdots,a_s/2\},2) \tag{3.3}$$

holds.

Proof. Suppose that there are vectors $\mathbf{x}_1,\cdots,\mathbf{x}_{s+1} \in \Lambda_4$ such that the equation (3.3) holds. We remark that the Minkowski reduction process does not raise the value of the diagonal elements (c.f. [37],[42]). Therefore the only possibility for that $(\mathfrak{T},\{a_1/2,\cdots,a_s/2\},2)$ is not 2-special is as follows. After the reduction the reduced matrix $U^t(\mathfrak{T},\{a_1/2,\cdots,a_s/2\},2)U$ contains 1 for its diagonal elements. This implies that the lattice $\mathcal{L}$ contains the vector $\mathbf{y}_1,\cdots,\mathbf{y}_{s+1}$ satisfying $[\mathbf{y}_1,\cdots,\mathbf{y}_{s+1}] = U^t(\mathfrak{T},\{a_1/2,\cdots,a_s/2\},2)U$, and consequently $\mathcal{L}$ contains a vector of norm 2. This contradicts to the extremality of $\mathcal{L}$. □

Lemma 3.5. *Suppose that* $(\mathfrak{T},\{a_1/2,\cdots,a_s/2\},2)$ *satisfies the same assumption as in Lemma* 3.4*, then*

$$\lambda_{a_1,\cdots,a_s}(2\mathfrak{T};\mathbf{x}_1,\cdots,\mathbf{x}_s) = 0.$$

where $[\mathbf{x}_1,\cdots,\mathbf{x}_s] = 2\mathfrak{T}, \mathbf{x}_1,\cdots,\mathbf{x}_s \in \Lambda_4$ *holds.*

Proof. This is an immediate consequence of Lemma 3.4. □

The pairs of integers $\langle a,b\rangle$ satisfying the conditions (i), (ii) are grouped into the sets according to the determinant of $2(\mathfrak{T}_{22},\{a/2,b/2\},2)$ and the equivalence by the unimodular transformations. In the ternary quadratic forms $\det(2(\mathfrak{T}_{22},\{a/2,b/2\},2))/2$ is called the discriminant of the matrix $T=(\mathfrak{T}_{22},\{a/2,b/2\},2)$ (c.f. Section 3.1). We denote it by d. We write $C_d(\mathfrak{T}_{22},\langle a,b\rangle)$ to denote the set of ordered pairs $\langle a',b'\rangle$ such that $\det(2(\mathfrak{T}_{22},\{a/2,b/2\},2))/2 = \det(2(\mathfrak{T}_{22},\{a'/2,b'/2\},2))/2=d$ and $(\mathfrak{T}_{22},\{a/2,b/2\},2)$ is equivalent to $(\mathfrak{T}_{22},\{a'/2,b'/2\},2)$. The total sets thus defined are called admissible sets with respect to $\mathfrak{T}_{22}$.

Later we will consider all possible pairs of integers a,b under the conditions (i) and (ii), but this time $\mathfrak{T}_{22}$ is replaced by $\mathfrak{T}_{21}$ or $\mathfrak{T}_{20}$. And we will consider admissible sets with respect to $\mathfrak{T}_{21}$ or $\mathfrak{T}_{20}$.

We set, for a fixed pair $\mathbf{x},\mathbf{y}\in\Lambda_4$ satisfying $(\mathbf{x},\mathbf{y})=2$,

$$\lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x},\mathbf{y}) = |\{\mathbf{z}\in\Lambda_4|(\mathbf{x},\mathbf{z})=a,(\mathbf{y},\mathbf{z})=b\}|.$$

The assumption (**MTA**) in this case implies the following.

Let $\langle a_1,b_1\rangle$ and $\langle a_2,b_2\rangle$ belong to the same set $C_d(\mathfrak{T}_{22},\langle a,b\rangle)$, then it holds that

$$\lambda_{a_1,b_1}(2\mathfrak{T}_{22};\mathbf{x},\mathbf{y}) = \lambda_{a_2,b_2}(2\mathfrak{T}_{22};\mathbf{x},\mathbf{y}). \tag{3.4}$$

We find that $C_{24}(\mathfrak{T}_{22},\langle 0,0\rangle)=\{\langle 0,0\rangle\}$ and

$$\begin{aligned}
C_{22}(\mathfrak{T}_{22},\langle 1,0\rangle) &= \{\langle 1,0\rangle,\langle 1,1\rangle,\langle 0,1\rangle,\langle -1,0\rangle,\langle -1,-1\rangle,\langle 0,-1\rangle\},\\
C_{18}(\mathfrak{T}_{22},\langle 2,1\rangle) &= \{\langle 2,1\rangle,\langle 1,-1\rangle,\langle 1,2\rangle,\langle -1,1\rangle,\langle -1,-2\rangle,\langle -2,-1\rangle\},\\
C_{16}(\mathfrak{T}_{22},\langle 2,0\rangle) &= \{\langle 2,0\rangle,\langle 2,2\rangle,\langle 0,2\rangle,\langle 0,-2\rangle,\langle -2,0\rangle,\langle -2,-2\rangle\},\\
C_{0}(\mathfrak{T}_{22},\langle 2,-2\rangle) &= \{\langle 2,-2\rangle,\langle -2,2\rangle,\langle 4,2\rangle,\langle -4,-2\rangle,\langle 2,4\rangle,\langle -2,4\rangle\}
\end{aligned}$$

exhaust all possible admissible sets. Let $\langle a,b\rangle$ be an element of one of the above admissible sets and $\lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x},\mathbf{y})$ be as above. We take $\boldsymbol{\alpha}=u\mathbf{x}+v\mathbf{y}$, where u and v are real independent variables, then we see that for $\mathbf{z}\in\Lambda_4$

$$(\mathbf{z},\boldsymbol{\alpha}) = au+bv,$$

holds for some a,b. One may note that $\lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x},\mathbf{y})=1$ for each ordered pair $\langle a,b\rangle\in C_0(2,-2)$ and that $(\boldsymbol{\alpha},\boldsymbol{\alpha})=4u^2+4uv+4v^2$. The relation (2.4) in our present case implies that ($\lambda_{a,b}=\lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x},\mathbf{y})$ for short)

(3.5)

$$\begin{aligned}
&\lambda_{1,0}\{(1u+0v)^2+((-1)u+0v)^2+(0u+1v)^2+(0u+(-1)v)^2+(1u+1v)^2+((-1)u+(-1)v)^2\}\\
&+\lambda_{2,1}\{(2u+1v)^2+((-2)u+(-1)v)^2+(1u+2v)^2+((-1)u+(-2)v)^2\\
&+(1u+(-1)v)^2+((-1)u+1v)^2\}\\
&+\lambda_{2,0}\{(2u+0v)^2+((-2)u+0v)^2+(2u+2v)^2+((-2)u+(-2)v)^2+(0u+2v)^2+(0u+(-2)v)^2\}\\
&+\{(2u-2v)^2+(-2u+2v)^2+(4u+2v)^2+((-4)u+(-2)v)^2+(2u+4v)^2+((-2)u+(-4)v)^2)\}\\
&=18360(4u^2+4uv+4v^2).
\end{aligned}$$

Since this equation is an identity for the polynomials in u,v, we must have an equation in the coefficients such as

$$4\lambda_{1,0}+12\lambda_{2,1}+16\lambda_{2,0}=73392. \tag{3.6}$$

From the relation (2.5) we have
(3.7)
$\lambda_{1,0}\{(1u+0v)^4+((-1)u+0v)^4+(0u+1v)^4+(0u+(-1)v)^4+(1u+1v)^4+((-1)u+(-1)v)^4\}$
$+\lambda_{2,1}\{(2u+1v)^4+((-2)u+(-1)v)^4+(1u+2v)^4+((-1)u+(-2)v)^4$
$+(1u+(-1)v)^4+((-1)u+1v)^4\}$
$+\lambda_{2,0}\{(2u+0v)^4+((-2)u+0v)^4+(2u+2v)^4+((-2)u+(-2)v)^4+(0u+2v)^4+(0u+(-2)v)^4\}$
$+\{(2u-2v)^4+(-2u+2v)^4+(4u+2v)^4+((-4)u+(-2)v)^4+(2u+4v)^4+((-2)u+(-4)v)^4)\}$
$=6480(4u^2+4uv+4v^2)^2.$

By this condition we have

(3.8) $$4\lambda_{1,0}+36\lambda_{2,1}+64\lambda_{2,0}=103104.$$

We have two other equations, but they give no new condition. Likewise from the relation (2.6) we get

(3.9) $$4\lambda_{1,0}+132\lambda_{2,1}+256\lambda_{2,0}=221952.$$

The linear equations (3.5), (3.8), (3.9) are enough to solve for $\lambda_{1,0},\lambda_{2,1},\lambda_{2,0}$. We find that

$$\lambda_{1,0}=14976,\lambda_{2,1}=896,\lambda_{2,0}=171.$$

We observe that the sum

$$\lambda_{0,0}+\sum_{\langle a,b\rangle\in C_{22}(1,0)}\lambda_{a,b}+\sum_{\langle a,b\rangle\in C_{18}(2,1)}\lambda_{a,b}+\sum_{\langle a,b\rangle\in C_{16}(2,0)}\lambda_{a,b}+\sum_{\langle a,b\rangle\in C_{0}(2,-2)}\lambda_{a,b}$$

must equal to the cardinality of $\Lambda_4=146880$, and consequently we have $\lambda_{0,0}=50616$.

3.4 A Justification of MTA in Degree 3 Case

Here we justify a weakened form of (**MTA**) for a special case.
Again we use $\lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x},\mathbf{y})$ for all elements $\langle a,b\rangle$ of the above admissible sets.

Then by using the equation (2.4) we have
(3.10)
$$\Bigg[\sum_{\langle a,b\rangle\in C_{22}(\mathfrak{T}_{22},\langle 1,0\rangle)}\lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)(au+bv)^2+\sum_{\langle a,b\rangle\in C_{18}(\mathfrak{T}_{22},\langle 2,1\rangle)}\lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)(au+bv)^2$$
$$+\sum_{\langle a,b\rangle\in C_{16}(\mathfrak{T}_{22},\langle 2,0\rangle)}\lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)(au+bv)^2+\sum_{\langle a,b\rangle\in C_{0}(\mathfrak{T}_{22},\langle 2,2\rangle)}\lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)(au+bv)^2\Bigg]$$
$$=18360(4u^2+4uv+4v^2).$$

This equation is a precise rewriting of Equation (3.5). From Equation (3.10) we have
(3.11)
$$\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}}\Bigg[\sum_{\langle a,b\rangle\in C_{22}(\mathfrak{T}_{22},\langle 1,0\rangle)}\lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)(au+bv)^2+\sum_{\langle a,b\rangle\in C_{18}(\mathfrak{T}_{22},\langle 2,1\rangle)}\lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)(au+bv)^2$$
$$+\sum_{\langle a,b\rangle\in C_{16}(\mathfrak{T}_{22},\langle 2,0\rangle)}\lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)(au+bv)^2+\sum_{\langle a,b\rangle\in C_{0}(\mathfrak{T}_{22},\langle 2,2\rangle)}\lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)(au+bv)^2\Bigg]$$
$$=\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}}18360(4u^2+4uv+4v^2).$$

From Equations (3.1) and (3.2) we have

$$\begin{aligned}&\sum_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\Lambda_4\\ [\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}}} \lambda_{a_1,b_1}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2) = a((\mathfrak{T}_{22},\{a_1/2,b_1/2\},2),\mathcal{L})\\ &= a((\mathfrak{T}_{22},\{a_2/2,b_2/2\},2),\mathcal{L}) = \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\Lambda_4\\ [\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}}} \lambda_{a_2,b_2}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2),\end{aligned}$$

for any two pairs $\langle a_1,b_1\rangle,\langle a_2,b_2\rangle \in C_{22}(\mathfrak{T}_{22},\langle 1,0\rangle)$ (resp. $C_{18}(\mathfrak{T}_{22},\langle 2,1\rangle)$,$C_{16}(\mathfrak{T}_{22},\langle 2,0\rangle)$). From these equalities we have that

$$\begin{aligned}&\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}} \sum_{\langle a,b\rangle\in C_{22}(\mathfrak{T}_{22},\langle 1,0\rangle)} \lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)(au+bv)^2\\ &= \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}} \lambda_{1,0}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)\sum_{\langle a,b\rangle\in C_{22}(\mathfrak{T}_{22},\langle 1,0\rangle)}(au+bv)^2,\\ &\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}} \sum_{\langle a,b\rangle\in C_{18}(\mathfrak{T}_{22},\langle 2,1\rangle)} \lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)(au+bv)^2\\ &= \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}} \lambda_{2,1}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)\sum_{\langle a,b\rangle\in C_{18}(\mathfrak{T}_{22},\langle 2,1\rangle)}(au+bv)^2,\\ &\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}} \sum_{\langle a,b\rangle\in C_{16}(\mathfrak{T}_{22},\langle 2,0\rangle)} \lambda_{a,b}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)(au+bv)^2\\ &= \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}} \lambda_{2,0}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)\sum_{\langle a,b\rangle\in C_{22}(\mathfrak{T}_{22},\langle 2,0\rangle)}(au+bv)^2.\end{aligned}$$

With the above equations, we may rewrite Equation (3.11) as

$$\begin{aligned}&\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}} \lambda_{1,0}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)\sum_{\langle a,b\rangle\in C_{22}(\mathfrak{T}_{22},\langle 1,0\rangle)}(au+bv)^2\\ &+\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}} \lambda_{2,1}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)\sum_{\langle a,b\rangle\in C_{18}(\mathfrak{T}_{22},\langle 2,1\rangle)}(au+bv)^2\\ &+\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}} \lambda_{2,0}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)\sum_{\langle a,b\rangle\in C_{16}(\mathfrak{T}_{22},\langle 2,0\rangle)}(au+bv)^2\\ &+\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}} \lambda_{2,2}(2\mathfrak{T}_{22};\mathbf{x}_1,\mathbf{x}_2)\sum_{\langle a,b\rangle\in C_{0}(\mathfrak{T}_{22},\langle 2,2\rangle)}(au+bv)^2\\ &=\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}} 18360(4u^2+4uv+4v^2),\end{aligned}$$

or

$$\begin{aligned}&a((\mathfrak{T}_{22},\{1/2,0/2\},2),\mathcal{L})\sum_{\langle a,b\rangle\in C_{22}(\mathfrak{T}_{22},\langle 1,0\rangle)}(au+bv)^2\\ &+a((\mathfrak{T}_{22},\{2/2,1/2\},2),\mathcal{L})\sum_{\langle a,b\rangle\in C_{18}(\mathfrak{T}_{22},\langle 2,1\rangle)}(au+bv)^2\\ &+a((\mathfrak{T}_{22},\{2/2,0/2\},2),\mathcal{L})\sum_{\langle a,b\rangle\in C_{16}(\mathfrak{T}_{22},\langle 2,0\rangle)}(au+bv)^2\\ &+\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}}\sum_{\langle a,b\rangle\in C_{0}(\mathfrak{T}_{22},\langle 2,2\rangle)}(au+bv)^2\\ &=\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}} 18360(4u^2+4uv+4v^2),\end{aligned}\tag{3.12}$$

Equation (3.5) can be regarded as a disguised form of Equation (3.12). Equation (3.7) can be regarded as a disguised form of Equation (3.13) below,

$$\begin{aligned}&a((\mathfrak{T}_{22},\{1/2,0/2\},2),\mathcal{L})\sum_{\langle a,b\rangle\in C_{22}(\mathfrak{T}_{22},\langle 1,0\rangle)}(au+bv)^4\\ &+a((\mathfrak{T}_{22},\{2/2,1/2\},2),\mathcal{L})\sum_{\langle a,b\rangle\in C_{18}(\mathfrak{T}_{22},\langle 2,1\rangle)}(au+bv)^4\\ &+a((\mathfrak{T}_{22},\{2/2,0/2\},2),\mathcal{L})\sum_{\langle a,b\rangle\in C_{16}(\mathfrak{T}_{22},\langle 2,0\rangle)}(au+bv)^4\\ &+\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}}\sum_{\langle a,b\rangle\in C_{0}(\mathfrak{T}_{22},\langle 2,2\rangle)}(au+bv)^4\\ &=\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}} 6480(4u^2+4uv+4v^2)^2,\end{aligned}\tag{3.13}$$

and there is another Equation (3.14):

$$\begin{aligned}
&a((\mathfrak{T}_{22},\{1/2,0/2\},2),\mathcal{L})\sum_{\langle a,b\rangle\in C_{22}(\mathfrak{T}_{22},\langle 1,0\rangle)}(au+bv)^6\\
&+a((\mathfrak{T}_{22},\{2/2,1/2\},2),\mathcal{L})\sum_{\langle a,b\rangle\in C_{18}(\mathfrak{T}_{22},\langle 2,1\rangle)}(au+bv)^6\\
&+a((\mathfrak{T}_{22},\{2/2,0/2\},2),\mathcal{L})\sum_{\langle a,b\rangle\in C_{16}(\mathfrak{T}_{22},\langle 2,0\rangle)}(au+bv)^6\\
&+\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}}\sum_{\langle a,b\rangle\in C_0(\mathfrak{T}_{22},\langle 2,2\rangle)}(au+bv)^6\\
&=\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}}3600(4u^2+4uv+4v^2)^3.
\end{aligned}\tag{3.14}$$

Equation (3.13) is obtained by using (2.5) and Equation (3.14) is obtained from (2.6). Equations (3.12),(3.13) and (3.14) are enough to determine the Fourier coefficients $a((\mathfrak{T}_{22},\{1/2,0/2\},2),\mathcal{L})$, $a((\mathfrak{T}_{22},\{2/2,1/2\},2),\mathcal{L})$, $a((\mathfrak{T}_{22},\{2/2,0/2\},2),\mathcal{L})$ explicitly.

3.5 Ternary Matrices Arising from $\mathfrak{T}_{21}$

As in the previous subsection we consider all possible pairs of integers $\langle a,b\rangle$ satisfying the conditions (i) and (ii) of the previous Subsubsection, but this time $\mathfrak{T}_{22}$ should be replaced by $\mathfrak{T}_{21}$ in the conditions. $C_d(\mathfrak{T}_{21},\langle a,b\rangle)$ has the similar meaning to that of the previous Subsection.

For a fixed pair $\mathbf{x},\mathbf{y}\in\Lambda_4$ satisfying $(\mathbf{x},\mathbf{y})=1$, we put

$$\lambda_{a,b}(2\mathfrak{T}_{21};\mathbf{x},\mathbf{y})=|\{\mathbf{z}\in\Lambda_4|(\mathbf{x},\mathbf{z})=a,(\mathbf{y},\mathbf{z})=b\}|.$$

The assumption (**MTA**) in this case implies the following.

Let $\langle a_1,b_1\rangle$ and $\langle a_2,b_2\rangle$ belong to the same set $C_d(\mathfrak{T}_{21},a,b)$, then it holds that

$$\lambda_{a_1,b_1}(2\mathfrak{T}_{21};\mathbf{x},\mathbf{y})=\lambda_{a_2,b_2}(2\mathfrak{T}_{21};\mathbf{x},\mathbf{y}).\tag{3.15}$$

We find that $C_{30}(\mathfrak{T}_{21},\langle 0,0\rangle)=\{\langle 0,0\rangle\}$ and

$$\begin{aligned}
&C_{28}(\mathfrak{T}_{21},\langle 1,0\rangle)=\{\langle 1,0\rangle,\langle 0,1\rangle,\langle -1,0\rangle,\langle 0,-1\rangle\},\\
&C_{27}(\mathfrak{T}_{21},\langle 1,1\rangle)=\{\langle 1,1\rangle,\langle -1,-1\rangle\},\\
&C_{25}(\mathfrak{T}_{21},\langle 1,-1\rangle)=\{\langle 1,-1\rangle,\langle -1,1\rangle\},\\
&C_{22}(\mathfrak{T}_{21},\langle 2,1\rangle)=\{\langle 2,1\rangle,\langle 2,0\rangle,\langle 1,2\rangle,\langle 0,2\rangle,\langle -2,-1\rangle,\langle -2,0\rangle,\langle -1,-2\rangle,\langle 0,-2\rangle\},\\
&C_{18}(\mathfrak{T}_{21},\langle 2,2\rangle)=\{\langle 2,2\rangle,\langle 2,-1\rangle,\langle 1,-2\rangle,\langle -2,-2\rangle,\langle -2,1\rangle,\langle -1,2\rangle\},\\
&C_{0}(\mathfrak{T}_{21},\langle 4,1\rangle)=\{\langle 4,1\rangle,\langle 1,4\rangle,\langle -4,-1\rangle,\langle -1,-4\rangle\}
\end{aligned}$$

exhaust all possible admissible sets with respect to $\mathfrak{T}_{21}$. Using $\boldsymbol{\alpha}=u\mathbf{x}+v\mathbf{y}$ with real variables u,v and noting that $(\boldsymbol{\alpha},\boldsymbol{\alpha})=4u^2+2uv+4v^2$, from (2.4) we have

$$\begin{aligned}
&\lambda_{1,0}\{(1x+0y)^2+((-1)x+0y)^2+(0x+1y)^2+(0x+(-1)y)^2\}+\\
&\lambda_{1,1}\{((1x+1y)^2+((-1)x+(-1)y)^2)\}+\\
&\lambda_{1,-1}\{(1x+(-1)y)^2+((-1)x+1y)^2\}+\\
&\lambda_{2,1}\{(2x+1y)^2+((-2)x+(-1)y)^2+(2x+0y)^2+((-2)x+0y)^2\\
&+(1x+2y)^2+((-1)x+(-2)y)^2+(0x+2y)^2+(0x+(-2)y)^2\}+\\
&\lambda_{2,2}\{(2x+2y)^2+((-2)x+(-2)y)^2+(2x+(-1)y)^2+((-2)x+1y)^2\\
&+(1x+(-2)y)^2+((-1)x+2y)^2\}+\\
&(4x+1y)^2+(1x+4y)^2+((-1)x+(-4)y)^2+((-4)x+(-1)y)^2\\
&=18360(4x^2+2xy+4y^2).
\end{aligned}$$

This is a polynomial identity with the variables x, y, and by comparing both sides we obtain

$$2\lambda_{1,0} + 2\lambda_{1,1} + 2\lambda_{1,-1} + 18\lambda_{2,1} + 18\lambda_{2,2} = 73406, \tag{3.16}$$

$$4\lambda_{1,1} - 4\lambda_{1,-1} + 16\lambda_{2,1} = 36688. \tag{3.17}$$

From (2.5) we obtain

$$\begin{aligned}
&\lambda_{1,0}\{(1x+0y)^4 + ((-1)x+0y)^4 + (0x+1y)^4 + (0x+(-1)y)^4\}+ \\
&\lambda_{1,1}\{((1x+1y)^4 + ((-1)x+(-1)y)^4)\}+ \\
&\lambda_{1,-1}\{(1x+(-1)y)^4 + ((-1)x+1y)^4\}+ \\
&\lambda_{2,1}\{(2x+1y)^4 + ((-2)x+(-1)y)^4 + (2x+0y)^4 + ((-2)x+0y)^4 \\
&+(1x+2y)^4 + ((-1)x+(-2)y)^4 + (0x+2y)^4 + (0x+(-2)y)^4\}+ \\
&\lambda_{2,2}\{(2x+2y)^4 + ((-2)x+(-2)y)^4 + (2x+(-1)y)^4 + ((-2)x+1y)^4 \\
&+(1x+(-2)y)^4 + ((-1)x+2y)^4\}+ \\
&(4x+1y)^4 + (1x+4y)^4 + ((-1)x+(-4)y)^4 + ((-4)x+(-1)y)^4 \\
&= 6480(4x^2+2xy+4y^2)^2.
\end{aligned}$$

By comparing the coefficients we have

$$2\lambda_{1,0} + 2\lambda_{1,1} + 2\lambda_{1,-1} + 66\lambda_{2,1} + 66\lambda_{2,2} = 103166, \tag{3.18}$$

$$8\lambda_{1,1} - 8\lambda_{1,-1} + 80\lambda_{2,1} + 48\lambda_{2,2} = 103136, \tag{3.19}$$

$$12\lambda_{1,1} + 12\lambda_{1,-1} + 96\lambda_{2,1} + 288\lambda_{2,2} = 232896. \tag{3.20}$$

From (2.6) we obtain

$$\begin{aligned}
&\lambda_{1,0}\{(1x+0y)^6 + ((-1)x+0y)^6 + (0x+1y)^6 + (0x+(-1)y)^6\}+ \\
&\lambda_{1,1}\{((1x+1y)^6 + ((-1)x+(-1)y)^6)\}+ \\
&\lambda_{1,-1}\{(1x+(-1)y)^6 + ((-1)x+1y)^6\}+ \\
&\lambda_{2,1}\{(2x+1y)^6 + ((-2)x+(-1)y)^6 + (2x+0y)^6 + ((-2)x+0y)^6 \\
&+(1x+2y)^6 + ((-1)x+(-2)y)^6 + (0x+2y)^6 + (0x+(-2)y)^6\}+ \\
&\lambda_{2,2}\{(2x+2y)^6 + ((-2)x+(-2)y)^6 + (2x+(-1)y)^6 + ((-2)x+1y)^6 \\
&+(1x+(-2)y)^6 + ((-1)x+2y)^6\}+ \\
&(4x+1y)^6 + (1x+4y)^6 + ((-1)x+(-4)y)^6 + ((-4)x+(-1)y)^6 \\
&= 6480(4x^2+2xy+4y^2)^3.
\end{aligned}$$

By comparing the coefficients we have

$$2\lambda_{1,0} + 2\lambda_{1,1} + 2\lambda_{1,-1} + 258\lambda_{2,1} + 258\lambda_{2,2} = 222206, \tag{3.21}$$

$$12\lambda_{1,1} - 12\lambda_{1,-1} + 408\lambda_{2,1} + 360\lambda_{2,2} = 333264, \tag{3.22}$$

$$30\lambda_{1,1} + 30\lambda_{1,-1} + 600\lambda_{2,1} + 2520\lambda_{2,2} = 855840, \tag{3.23}$$

$$40\lambda_{1,1} - 40 * \lambda_{1,-1} + 640\lambda_{2,1} + 1920\lambda_{2,2} = 714880. \tag{3.24}$$

Thus we have plenty of linear equations (3.8)$\sim$(3.16) on λ's. By solving them we find

$$\begin{aligned}
&\lambda_{1,0} = \lambda_{0,1} = \lambda_{-1,0} = \lambda_{0,-1} = 17235,\\
&\lambda_{1,1} = \lambda_{-1,-1} = 10360,\\
&\lambda_{1,-1} = \lambda_{-1,1} = 3528,\\
&\lambda_{2,1} = \lambda_{2,0} = \lambda_{1,2} = \lambda_{0,2} = \lambda_{-2,-1} = \lambda_{-2,0} = \lambda_{-1,-2} = \lambda_{0,-2} = 585,\\
&\lambda_{2,2} = \lambda_{2,-1} = \lambda_{1,-2} = \lambda_{-2,-2} = \lambda_{-2,1} = \lambda_{-1,2} = 35.
\end{aligned}$$

As to $\lambda_{0,0}$ the following relation, whose reason is similar to that in the last part of the preceding subsection,

$$\begin{aligned}
&\lambda_{0,0} + \textstyle\sum_{\langle a,b\rangle \in C_{28}(1,0)} \lambda_{a,b} + \sum_{\langle a,b\rangle \in C_{27}(1,1)} \lambda_{a,b} + \sum_{\langle a,b\rangle \in C_{25}(1,-1)} \lambda_{a,b}\\
&+ \textstyle\sum_{\langle a,b\rangle \in C_{22}(2,1)} \lambda_{a,b} + \sum_{\langle a,b\rangle \in C_{18}(2,2)} \lambda_{a,b} + \sum_{\langle a,b\rangle \in C_{0}(4,1)} \lambda_{a,b}
\end{aligned}$$

must equal to 146880, and we have $\lambda_{0,0} = 45270$.

3.6 Concerning Ternary Matrices Arising from $\mathfrak{T}_{20}$

For a fixed pair $\mathbf{x}, \mathbf{y} \in \Lambda_4$ satisfying $(\mathbf{x}, \mathbf{y}) = 0$, we put

$$\lambda_{a,b}(2\mathfrak{T}_{20}; \mathbf{x}, \mathbf{y}) = |\{\mathbf{z} \in \Lambda_4 | (\mathbf{x}, \mathbf{z}) = a, (\mathbf{y}, \mathbf{z}) = b\}|.$$

As before we use the shortened notation: $\lambda_{a,b} = \lambda_{a,b}(2\mathfrak{T}_{20}; \mathbf{x}, \mathbf{y})$. The set $C_{30}(\mathfrak{T}_{20}, \langle 0,0\rangle) = \{\langle 0,0\rangle\}$ and the sets

$$\begin{aligned}
&C_{30}(\mathfrak{T}_{20}, \langle 1,0\rangle) = \{\langle 1,0\rangle, \langle 0,1\rangle, \langle -1,0\rangle, \langle 0,-1\rangle\},\\
&C_{28}(\mathfrak{T}_{20}, \langle 1,1\rangle) = \{\langle 1,1\rangle, \langle 1,-1\rangle, \langle -1,-1\rangle, \langle -1,1\rangle\},\\
&C_{24}(\mathfrak{T}_{20}, \langle 2,0\rangle) = \{\langle 2,0\rangle, \langle 0,2\rangle, \langle -2,0\rangle, \langle 0,-2\rangle\},\\
&C_{22}(\mathfrak{T}_{20}, \langle 2,1\rangle) = \{\langle 2,1\rangle, \langle 2,-1\rangle, \langle 1,2\rangle, \langle 1,-2\rangle, \langle -2,-1\rangle, \langle -2,1\rangle, \langle -1,-2\rangle, \langle -1,2\rangle\},\\
&C_{16}(\mathfrak{T}_{20}, \langle 2,2\rangle) = \{\langle 2,2\rangle, \langle 2,-2\rangle, \langle -2,-2\rangle, \langle -2,2\rangle\},\\
&C_{0}(\mathfrak{T}_{20}, \langle 4,0\rangle) = \{\langle 4,0\rangle, \langle 0,4\rangle, \langle -4,0\rangle, \langle 0,-4\rangle\}
\end{aligned}$$

exhaust all possible admissible sets with respect to $\mathfrak{T}_{20}$. Using $\boldsymbol{\alpha} = u\mathbf{x}+v\mathbf{y}$ with real variables u, v and noting that $(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = 4u^2 + 4v^2$, from (2.4) $\sim$(2.7) we have the relations:

$$\begin{aligned}
&\lambda_{0,0} = 43740 + 36\lambda_{2,2},\\
&\lambda_{1,0} = 17824 - 24\lambda_{2,2},\\
&\lambda_{1,1} = -6720 + 16\lambda_{2,2},\\
&\lambda_{2,0} = 760 + 6\lambda_{2,2},\\
&\lambda_{2,1} = 240 - 4\lambda_{2,2}.
\end{aligned}$$

We can not determine λ's more precisely. However we can compute the Fourier coefficients $a(T, \mathcal{L}_{32})$ of Siegel theta series of degree 3 associated for the indices:

$$T = (\mathfrak{T}_{20}, \{a/2, b/2\}, 2), \langle a,b\rangle = \langle 0,0\rangle, \langle 1,0\rangle, \langle 1,1\rangle, \langle 2,0\rangle, \langle 2,1\rangle$$

by other bypasses.

3.7 A Small Table

Here we give a table of the Fourier coefficients of Siegel theta series of degree 3. The index T of size 3 is given by the ordered sextuple $[t_{11}, t_{22}, t_{33}, t_{12}, t_{13}, t_{23}]$.

Table 4. Fourier coefficients of Siegel theta series of degree 3 for the 32-dimensional extremal lattice

D	T	*value*
*16	$[2,2,2,0,2,2]$	31144435200
18	$[2,2,2,-1,1,2]$	163189555200
22	$[2,2,2,0,1,2]$	2727596851200
*24	$[2,2,2,0,0,2]$	9218752819200
25	$[2,2,2,-1,1,1]$	16449507164160
27	$[2,2,2,1,1,1]$	48304108339200
28	$[2,2,2,0,1,1]$	80359199539200
30	$[2,2,2,0,0,1]$	211074033254400
*32	$[2,2,2,0,0,0]$	520930019059200

4 Siegel Theta Series of Degree 4

4.1 A Trial to Compute Fourier Coefficients of Siegel Theta Series of Degree 4

We name

$$\mathfrak{T}_{30} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \ \mathfrak{T}_{31} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1/2 \\ 1 & 1/2 & 2 \end{pmatrix}.$$

We use the symbol $(\mathfrak{T}_{3*}, \{a/2, b/2, c/2\}, 2)$ to denote the matrix

$$\begin{pmatrix} t_{11} & t_{12}/2 & t_{13}/2 & a/2 \\ t_{12}/2 & t_{22} & t_{23}/2 & b/2 \\ t_{13}/2 & t_{23}/2 & t_{33} & c/2 \\ a/2 & b/2 & c/2 & 2 \end{pmatrix}, \tag{4.1}$$

where

$$\mathfrak{T}_{3*} = \begin{pmatrix} t_{11} & t_{12}/2 & t_{13}/2 \\ t_{12}/2 & t_{22} & t_{23}/2 \\ t_{13}/2 & t_{23}/2 & t_{33} \end{pmatrix}. \tag{4.2}$$

4.1.1 Extending $\mathfrak{T}_{30}$

Now we seek all ordered triples $\langle a, b, c\rangle$ of integers satisfying the two conditions
(1) $(\mathfrak{T}_{30}, \{a/2, b/2, c/2\}, 2)$ is positive semi-definite,
(2) when $(\mathfrak{T}_{30}, \{a/2, b/2, c/2\}, 2)$ is reduced under unimodular transformations: $U^t(\mathfrak{T}_{30}, \{a/2, b/2, c/2\}, 2)U$, the minimal value of the non-zero diagonal entries of the resulting matrix is 2.

We remark that it is easy to find all possible ordered triples $\langle a,b,c\rangle$ satisfying the condition (1) only. To eliminate the triples $\langle a,b,c\rangle$ not satisfying the condition (2) we use the theta series of one complex variable associated with the quadratic form defined by $(\mathfrak{T},\{a/2,b/2,c/2\},2)$. If $(\mathfrak{T},\{a/2,b/2,c/2\},2)$ integrally represents 1, then $(\mathfrak{T}_{30},\{a/2,b/2,c/2\},2)$ does not satisfy the condition (2). At this stage we do not know whether two triples that lead to the quaternary forms with the identical discriminant belong to the same set $(\mathfrak{T}_{30},\{a/2,b/2,c/2\},2)$. We consult the table in [19]. In viewing the table we observe that in the cases when $d=116,112,96,84$ there is a unique reduced quaternary form that has diagonal entries all 2. The table [19] does not give non-primitive forms. In the cases when $d=128,80,64$ there arises a unique non-primitive form by multiplying each coefficient of the unique reduced form of discriminant $d=8$ (resp. $d=5,4$) by 2. We verify that each member $\langle a,b,c\rangle\in C_d(\mathfrak{T}_{30},\langle a,b,c\rangle)$ gives rise to an identical reduced form.

Before giving a summary we use a convention: $C_d(\mathfrak{T}_{30},\langle a,b,c\rangle)=C_d^{(+)}(\mathfrak{T}_{30},\langle a,b,c\rangle)\cup C_d^{(-)}(\mathfrak{T}_{30},\langle a,b,c\rangle)$, where $C_d^{(-)}(\mathfrak{T}_{30},\langle a,b,c\rangle)$ is obtained by

$$C_d^{(-)}(\mathfrak{T}_{30},\langle a,b,c\rangle)=\{\langle -a,-b,-c\rangle\ |\langle a,b,c\rangle\in C_d^{(+)}(\mathfrak{T}_{30},\langle a,b,c\rangle)\}. \tag{4.1}$$

This convention is clarified by the following explicit presentations of $C_d^{(+)}(\mathfrak{T}_{30},\langle a,b,c\rangle)$'s.

We summarize our search by concluding that $C_{128}(\mathfrak{T}_{30},\langle 0,0,0\rangle)=\{\langle 0,0,0\rangle\}$ and

$$\begin{aligned}
C_{116}^{(+)}(\mathfrak{T}_{30},\langle 1,1,1\rangle)&=\{\langle 1,1,1\rangle,\langle 0,0,1\rangle,\langle 0,1,0\rangle,\langle 1,0,0\rangle\},\\
C_{112}^{(+)}(\mathfrak{T}_{30},\langle 1,1,0\rangle)&=\{\langle 1,1,0\rangle,\langle 1,0,1\rangle,\langle 0,1,1\rangle\},\\
C_{96}^{(+)}(\mathfrak{T}_{30},\langle 2,1,1\rangle)&=\{\langle 2,1,1\rangle,\langle 1,2,1\rangle,\langle 1,1,2\rangle,\langle 1,-1,0\rangle,\langle 0,1,-1\rangle,\langle 1,0,-1\rangle\},\\
C_{84}^{(+)}(\mathfrak{T}_{30},\langle 2,2,1\rangle)&=\{\langle 2,2,1\rangle,\langle 2,1,2\rangle,\langle 1,2,2\rangle,\langle 2,1,0\rangle,\langle 2,0,1\rangle,\langle 1,2,0\rangle,\\
&\qquad\langle 1,0,2\rangle,\langle 0,2,1\rangle,\langle 0,1,2\rangle,\langle 1,1,-1\rangle,\langle 1,-1,1\rangle,\langle -1,1,1\rangle\},\\
C_{80}^{(+)}(\mathfrak{T}_{30},\langle 2,2,2\rangle)&=\{\langle 2,2,2\rangle,\langle 2,0,0\rangle,\langle 0,2,0\rangle,\langle 0,0,2\rangle\},\\
C_{64}^{(+)}(\mathfrak{T}_{30},\langle 2,2,0\rangle)&=\{\langle 2,2,0\rangle,\langle 2,0,2\rangle,\langle 0,2,2\rangle\},\\
C_{0}^{(+)}(\mathfrak{T}_{30},\langle 4,2,2\rangle)&=\{\langle 4,2,2\rangle,\langle 2,4,2\rangle,\langle 2,2,4\rangle,\langle 2,0,-2\rangle,\langle 2,-2,0\rangle,\langle 0,2,-2\rangle\},
\end{aligned}$$

exhaust all admissible sets for $\mathfrak{T}_{30}$. We put

$$\Lambda_4^3(2\mathfrak{T}_{30})=\{(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)\in\Lambda_4^3|\ [\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\}.$$

For a triple $(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)\in\Lambda_4^3(2\mathfrak{T}_{30})$ and for an ordered triple $\langle a,b,c\rangle$ in one of the above sets we put

$$L_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)=\{\mathbf{z}\in\Lambda_4|(\mathbf{x}_1,\mathbf{z})=a,(\mathbf{x}_2,\mathbf{z})=b,(\mathbf{x}_1,\mathbf{x}_2)=c\}, \tag{4.4}$$

$$\lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)=|L_{a,b,c}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)|. \tag{4.5}$$

We use the formulas (2.4) $\sim$ (2.7). We set $\boldsymbol{\alpha}=u\mathbf{x}_1+v\mathbf{x}_2+w\mathbf{x}_3$, where u,v,w are real independent variables and $\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4$ satisfying $[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}$. Then it holds that

$(\boldsymbol{\alpha},\boldsymbol{\alpha}) = 4u^2+4v^2+4w^2+4uv+4uw+4vw$. The left-hand side of (2.4) is

(4.6)

$$\begin{aligned}
&\sum_{\mathbf{z}\in\Lambda_4}(\mathbf{z},\boldsymbol{\alpha})^2 \\
&= \sum_{\langle a,b,c\rangle\in C_{116}(\mathfrak{T}_{30},\langle 1,1,1\rangle)} \lambda_{a,b,c}(au+bv+cw)^2 + \sum_{\langle a,b,c\rangle\in C_{112}(\mathfrak{T}_{30},\langle 1,1,0\rangle)} \lambda_{a,b,c}(au+bv+cw)^2 \\
&\quad + \sum_{\langle a,b,c\rangle\in C_{96}(\mathfrak{T}_{30},\langle 2,1,1\rangle)} \lambda_{a,b,c}(au+bv+cw)^2 + \sum_{\langle a,b,c\rangle\in C_{84}(\mathfrak{T}_{30},\langle 2,2,1\rangle)} \lambda_{a,b,c}(au+bv+cw)^2 \\
&\quad + \sum_{\langle a,b,c\rangle\in C_{80}(\mathfrak{T}_{30},\langle 2,2,2\rangle)} \lambda_{a,b,c}(au+bv+cw)^2 + \sum_{\langle a,b,c\rangle\in C_{64}(\mathfrak{T}_{30},\langle 2,2,0\rangle)} \lambda_{a,b,c}(au+bv+cw)^2 \\
&\quad + \sum_{\langle a,b,c\rangle\in C_{0}(\mathfrak{T}_{30},\langle 4,2,2\rangle)} (au+bv+cw)^2 \\
&= 2\lambda_{1,1,1}((u+v+w)^2+w^2+v^2+u^2)+2\lambda_{1,1,0}((u+v)^2+(u+w)^2+(v+w)^2) \\
&\quad +2\lambda_{2,1,1}((2u+v+w)^2+(u+2v+w)^2+(u+v+2w)^2) \\
&\quad +2\lambda_{2,2,1}((2u+2v+w)^2+(2u+v+2w)^2+(u+2v+2w)^2 \\
&\quad +(2u+v)^2+(2u+w)^2+(u+2v)^2+(u+2w)^2+(2v+w)^2+(v+2w)^2 \\
&\quad +(u+v-w)^2+(u-v+w)^2+(-u+v+w)^2) \\
&\quad +2\lambda_{2,2,2}((2u+2v+2w)^2+(2w)^2+(2v)^2+(2u)^2) \\
&\quad +2\lambda_{2,2,0}((2u+2v)^2+(2u+2w)^2+(2v+2w)^2) \\
&\quad +2((4u+2v+2w)^2+(2u+4v+2w)^2+(2u+2v+4w)^2+(2u-2v)^2+(2u-2w)^2+(2v-2w)^2) \\
&= 18360(4u^2+4v^2+4w^2+4uv+4uw+4vw) \text{ (the right-hand side of (2.4)).}
\end{aligned}$$

From (2.5) we obtain

$$\begin{aligned}
&2\lambda_{1,1,1}((u+v+w)^4+w^2+v^2+u^2)+2\lambda_{1,1,0}((u+v)^4+(u+w)^4+(v+w)^4) \\
&\quad +2\lambda_{2,1,1}((2u+v+w)^4+(u+2v+w)^4+(u+v+2w)^4) \\
&\quad +2\lambda_{2,2,1}((2u+2v+w)^4+(2u+v+2w)^4+(u+2v+2w)^4 \\
&\quad +(2u+v)^4+(2u+w)^4+(u+2v)^4+(u+2w)^4+(2v+w)^4+(v+2w)^4 \\
&\quad +(u+v-w)^4+(u-v+w)^4+(-u+v+w)^4) \\
&\quad +2\lambda_{2,2,2}((2u+2v+2w)^4+(2w)^4+(2v)^4+(2u)^4) \\
&\quad +2\lambda_{2,2,0}((2u+2v)^4+(2u+2w)^4+(2v+2w)^4) \\
&\quad +2((4u+2v+2w)^4+(2u+4v+2w)^4+(2u+2v+4w)^4+(2u-2v)^4+(2u-2w)^4+(2v-2w)^4) \\
&= 6480(4u^2+4v^2+4w^2+4uv+4uw+4vw)^2.
\end{aligned}$$

From (2.6) we obtain

$$\begin{aligned}
&2\lambda_{1,1,1}((u+v+w)^6+w^2+v^2+u^2)+2\lambda_{1,1,0}((u+v)^6+(u+w)^6+(v+w)^6)\\
&+2\lambda_{2,1,1}((2u+v+w)^6+(u+2v+w)^6+(u+v+2w)^6)\\
&+2\lambda_{2,2,1}((2u+2v+w)^6+(2u+v+2w)^6+(u+2v+2w)^6\\
&+(2u+v)^6+(2u+w)^6+(u+2v)^6+(u+2w)^6+(2v+w)^6+(v+2w)^6\\
&+(u+v-w)^6+(u-v+w)^6+(-u+v+w)^6)\\
&+2\lambda_{2,2,2}((2u+2v+2w)^6+(2w)^6+(2v)^6+(2u)^6)\\
&+2\lambda_{2,2,0}((2u+2v)^6+(2u+2w)^6+(2v+2w)^6)\\
&+2((4u+2v+2w)^6+(2u+4v+2w)^6+(2u+2v+4w)^6+(2u-2v)^6+(2u-2w)^6+(2v-2w)^6)\\
=\;&3600(4u^2+4v^2+4w^2+4uv+4uw+4vw)^3.
\end{aligned}$$

The above three equations are polynomial identities with the variables u, v, w. By comparing the coefficients we get many linear equations on λ's. We can not solve these equation, but we obtain the following expressions:

$$(4.7)\quad \begin{cases}
\lambda_{1,1,1}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = 9312-10\lambda_{2,2,2}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3),\\
\lambda_{1,1,0}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = 4992+8\lambda_{2,2,2}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3),\\
\lambda_{2,1,1}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = 448+4\lambda_{2,2,2}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3),\\
\lambda_{2,2,1}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = 224-2\lambda_{2,2,2}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3),\\
\lambda_{2,2,2}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = \lambda_{2,2,2}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3),\\
\lambda_{2,2,0}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = \lambda_{2,2,2}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)-54.
\end{cases}$$

As to the quantity $\lambda_{0,0,0}$ we use the following equation:

$$\begin{aligned}
&\lambda_{0,0,0}+\sum_{\langle a,b,c\rangle\in C_{116}(\mathfrak{T}_{30},\langle 1,1,1\rangle)}\lambda_{a,b,c}+\sum_{\langle a,b,c\rangle\in C_{112}(\mathfrak{T}_{30},\langle 1,1,0\rangle)}\lambda_{a,b,c}\\
&+\sum_{\langle a,b,c\rangle\in C_{96}(\mathfrak{T}_{30},\langle 2,1,1\rangle)}\lambda_{a,b,c}+\sum_{\langle a,b,c\rangle\in C_{84}(\mathfrak{T}_{30},\langle 2,2,1\rangle)}\lambda_{a,b,c}\\
&+\sum_{\langle a,b,c\rangle\in C_{80}(\mathfrak{T}_{30},\langle 2,2,2\rangle)}\lambda_{a,b,c}+\sum_{\langle a,b,c\rangle\in C_{64}(\mathfrak{T}_{30},\langle 2,2,0\rangle)}\lambda_{a,b,c}\\
&+\sum_{\langle a,b,c\rangle\in C_{0}(\mathfrak{T}_{30},\langle 4,2,2\rangle)}1\\
=\;&146880,
\end{aligned}$$

which has the quite similar reason as the argument right after the equation (3.7). Consequently we get

$$\lambda_{0,0,0}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)=31992+18\lambda_{2,2,2}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3).$$

We remark that the equation (2.7) does not give any further condition in determining λ's more precisely. At best we prove the following

Proposition 4.1. *For a 32-dimensional even unimodular extremal lattice $\mathcal{L}_{32}$ we have the* *following Fourier coefficients relations:*

(4.8)

$$\begin{aligned}
a((\mathfrak{T}_{30},\{0,0,0\},2),\mathcal{L}_{32}) &= 31992\cdot 31144435200+18\cdot a((\mathfrak{T}_{30},\{1,1,1\},2),\mathcal{L}_{32}),\\
a((\mathfrak{T}_{30},\{1/2,1/2,1/2\},2),\mathcal{L}_{32}) &= 9312\cdot 31144435200-10\cdot a((\mathfrak{T}_{30},\{1,1,1\},2),\mathcal{L}_{32}),\\
a((\mathfrak{T}_{30},\{1/2,1/2,0\},2),\mathcal{L}_{32}) &= 4992\cdot 31144435200+8\cdot a((\mathfrak{T}_{30},\{1,1,1\},2),\mathcal{L}_{32}),\\
a((\mathfrak{T}_{30},\{1,1/2,1/2\},2),\mathcal{L}_{32}) &= 448\cdot 31144435200+4\cdot a((\mathfrak{T}_{30},\{1,1,1\},2),\mathcal{L}_{32}),\\
a((\mathfrak{T}_{30},\{1,1,1/2\},2),\mathcal{L}_{32}) &= 224\cdot 31144435200-2\cdot a((\mathfrak{T}_{30},\{1,1,1\},2),\mathcal{L}_{32}),\\
a((\mathfrak{T}_{30},\{1,1,0\},2),\mathcal{L}_{32}) &= -54\cdot 31144435200+a((\mathfrak{T}_{30},\{1,1,1\},2),\mathcal{L}_{32}).
\end{aligned}$$

Proof. We specify the formula (3.1) to $a((\mathfrak{T}_{30},\{0,0,0\},2),\mathcal{L}_{32})$ and we obtain

$$\begin{aligned}
a((\mathfrak{T}_{30},\{0,0,0\},2),\mathcal{L}_{32}) &= \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4\\ [\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}}} \lambda_{0,0,0}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)\\
&= \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4\\ [\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}}} (31992+18\cdot\lambda_{2,2,2}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3))\\
&= 31992\cdot 31144435200+18\cdot a((\mathfrak{T}_{30},\{1,1,1\},2),\mathcal{L}_{32}).
\end{aligned}$$

This proves the first formula of Proposition. Other formulas in the proposition are proved by using the relations (4.7). □

4.1.2 A Brief Justification of MTA in Degree 4 Case

In this section we derive linear constraints on the quantities $a((\mathfrak{T}_{30},\{1,1,0\},2),\mathcal{L}_{32}),\cdots,$ $a((\mathfrak{T}_{30},\{2,2,0\},2),\mathcal{L}_{32})$ without appealing to **MTA**. The resulting conclusion is the same as that of using **MTA**. This justifies using of **MTA** in determining the relations among the Fourier coefficients.

Let $\lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$ be as before. We begin with the precise writing of (4.6):

$$
\begin{aligned}
&(4.9)\\
&\sum_{\mathbf{z}\in\Lambda_4}(\mathbf{z},\boldsymbol{\alpha})^2\\
&= \sum_{\langle a,b,c\rangle\in C_{116}(\mathfrak{T}_{30},\langle 1,1,1\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2\\
&+ \sum_{\langle a,b,c\rangle\in C_{112}(\mathfrak{T}_{30},\langle 1,1,0\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2\\
&+ \sum_{\langle a,b,c\rangle\in C_{96}(\mathfrak{T}_{30},\langle 2,1,1\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2\\
&+ \sum_{\langle a,b,c\rangle\in C_{84}(\mathfrak{T}_{30},\langle 2,2,1\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2\\
&+ \sum_{\langle a,b,c\rangle\in C_{80}(\mathfrak{T}_{30},\langle 2,2,2\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2\\
&+ \sum_{\langle a,b,c\rangle\in C_{64}(\mathfrak{T}_{30},\langle 2,2,0\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2\\
&+ \sum_{\langle a,b,c\rangle\in C_{0}(\mathfrak{T}_{30},\langle 4,2,2\rangle)} (au+bv+cw)^2\\
&= 18360(4u^2+4v^2+4w^2+4uv+4uw+2vw).
\end{aligned}
$$

From (4.9) we have

(4.10)

$$\sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \left[\sum_{\mathbf{z}\in\Lambda_4}(\mathbf{z},\boldsymbol{\alpha})^2\right]$$

$$= \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \left[\sum_{\langle a,b,c\rangle\in C_{116}(\mathfrak{T}_{30},\langle 1,1,1\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2\right.$$

$$+ \sum_{\langle a,b,c\rangle\in C_{112}(\mathfrak{T}_{30},\langle 1,1,0\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2$$

$$+ \sum_{\langle a,b,c\rangle\in C_{96}(\mathfrak{T}_{30},\langle 2,1,1\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2$$

$$+ \sum_{\langle a,b,c\rangle\in C_{84}(\mathfrak{T}_{30},\langle 2,2,1\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2$$

$$+ \sum_{\langle a,b,c\rangle\in C_{80}(\mathfrak{T}_{30},\langle 2,2,2\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2$$

$$+ \sum_{\langle a,b,c\rangle\in C_{64}(\mathfrak{T}_{30},\langle 2,2,0\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2$$

$$\left.+ \sum_{\langle a,b,c\rangle\in C_{0}(\mathfrak{T}_{30},\langle 4,2,2\rangle)} (au+bv+cw)^2\right]$$

$$= \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2]=2\mathfrak{T}_{22}\\ \mathbf{x}_1,\mathbf{x}_2\in\Lambda_4}} 18360(4u^2+4v^2+4w^2+4uv+4uw+2vw).$$

The second term in Eq. (4.10) is rewritten as

$$
\begin{aligned}
& \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \sum_{\langle a,b,c\rangle\in C_{116}(\mathfrak{T}_{30},\langle 1,1,1\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2 \\
& + \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \sum_{\langle a,b,c\rangle\in C_{112}(\mathfrak{T}_{30},\langle 1,1,0\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2 \\
& + \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \sum_{\langle a,b,c\rangle\in C_{96}(\mathfrak{T}_{30},\langle 2,1,1\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2 \\
& + \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \sum_{\langle a,b,c\rangle\in C_{84}(\mathfrak{T}_{30},\langle 2,2,1\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2 \\
& + \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \sum_{\langle a,b,c\rangle\in C_{80}(\mathfrak{T}_{30},\langle 2,2,2\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2 \\
& + \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \sum_{\langle a,b,c\rangle\in C_{64}(\mathfrak{T}_{30},\langle 2,2,0\rangle)} \lambda_{a,b,c}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+cw)^2 \\
& + \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \sum_{\langle a,b,c\rangle\in C_{0}(\mathfrak{T}_{30},\langle 4,2,2\rangle)} (au+bv+cw)^2 \\
= & \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \lambda_{1,1,1}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \sum_{\langle a,b,c\rangle\in C_{116}(\mathfrak{T}_{30},\langle 1,1,1\rangle)} (au+bv+cw)^2 \\
& + \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \lambda_{1,1,0}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \sum_{\langle a,b,c\rangle\in C_{112}(\mathfrak{T}_{30},\langle 1,1,0\rangle)} (au+bv+cw)^2 \\
& + \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \lambda_{2,1,1}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \sum_{\langle a,b,c\rangle\in C_{96}(\mathfrak{T}_{30},\langle 2,1,1\rangle)} (au+bv+cw)^2 \\
& + \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \lambda_{2,2,1}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \sum_{\langle a,b,c\rangle\in C_{84}(\mathfrak{T}_{30},\langle 2,2,1\rangle)} (au+bv+cw)^2 \\
& + \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \lambda_{2,2,2}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \sum_{\langle a,b,c\rangle\in C_{80}(\mathfrak{T}_{30},\langle 2,2,2\rangle)} (au+bv+cw)^2 \\
& + \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \lambda_{2,2,0}(2\mathfrak{T}_{30};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \sum_{\langle a,b,c\rangle\in C_{64}(\mathfrak{T}_{30},\langle 2,2,0\rangle)} (au+bv+cw)^2 \\
& + \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30}\\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \sum_{\langle a,b,c\rangle\in C_{0}(\mathfrak{T}_{30},\langle 4,2,2\rangle)} (au+bv+cw)^2 \\
= & \; a((\mathfrak{T}_{30},\{1,1,1\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{116}(\mathfrak{T}_{30},\langle 1,1,1\rangle)} (au+bv+cw)^2
\end{aligned}
$$

$$
\begin{aligned}
&+a((\mathfrak{T}_{30},\{1,1,0\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{112}(\mathfrak{T}_{30},\langle 1,1,0\rangle)} (au+bv+cw)^2 \\
&+a((\mathfrak{T}_{30},\{2,1,1\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{96}(\mathfrak{T}_{30},\langle 2,1,1\rangle)} (au+bv+cw)^2 \\
&+a((\mathfrak{T}_{30},\{2,2,1\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{84}(\mathfrak{T}_{30},\langle 2,2,1\rangle)} (au+bv+cw)^2 \\
&+a((\mathfrak{T}_{30},\{2,2,2\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{80}(\mathfrak{T}_{30},\langle 2,2,2\rangle)} (au+bv+cw)^2 \\
&+a((\mathfrak{T}_{30},\{2,2,0\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{64}(\mathfrak{T}_{30},\langle 2,2,0\rangle)} (au+bv+cw)^2 \\
&+a(\mathfrak{T}_{30},\mathcal{L}_{32}) \sum_{\substack{[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{30} \\ \mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4}} \sum_{\langle a,b,c\rangle\in C_0(\mathfrak{T}_{30},\langle 4,2,2\rangle)} (au+bv+cw)^2
\end{aligned}
$$

Concludingly we have

(4.11)

$$
\begin{aligned}
&a((\mathfrak{T}_{30},\{1,1,0\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{112}(\mathfrak{T}_{30},\langle 1,1,0\rangle)} (au+bv+cw)^2 \\
&+a((\mathfrak{T}_{30},\{2,1,1\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{96}(\mathfrak{T}_{30},\langle 2,1,1\rangle)} (au+bv+cw)^2 \\
&+a((\mathfrak{T}_{30},\{2,2,1\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{84}(\mathfrak{T}_{30},\langle 2,2,1\rangle)} (au+bv+cw)^2 \\
&+a((\mathfrak{T}_{30},\{2,2,2\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{80}(\mathfrak{T}_{30},\langle 2,2,2\rangle)} (au+bv+cw)^2 \\
&+a((\mathfrak{T}_{30},\{2,2,0\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{64}(\mathfrak{T}_{30},\langle 2,2,0\rangle)} (au+bv+cw)^2 \\
&+a(\mathfrak{T}_{30},\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_0(\mathfrak{T}_{30},\langle 4,2,2\rangle)} (au+bv+cw)^2 \\
=\;& a(\mathfrak{T}_{30},\mathcal{L}_{32})\cdot 18360(4u^2+4v^2+4w^2+4uv+4uw+2vw)
\end{aligned}
$$

Likewise by using (2.5) and (2.6) we have

$$(4.12)$$

$$\begin{aligned}
& a((\mathfrak{T}_{30},\{1,1,0\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{112}(\mathfrak{T}_{30},\langle 1,1,0\rangle)} (au+bv+cw)^4 \\
& +a((\mathfrak{T}_{30},\{2,1,1\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{96}(\mathfrak{T}_{30},\langle 2,1,1\rangle)} (au+bv+cw)^4 \\
& +a((\mathfrak{T}_{30},\{2,2,1\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{84}(\mathfrak{T}_{30},\langle 2,2,1\rangle)} (au+bv+cw)^4 \\
& +a((\mathfrak{T}_{30},\{2,2,2\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{80}(\mathfrak{T}_{30},\langle 2,2,2\rangle)} (au+bv+cw)^4 \\
& +a((\mathfrak{T}_{30},\{2,2,0\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{64}(\mathfrak{T}_{30},\langle 2,2,0\rangle)} (au+bv+cw)^4 \\
& +a(\mathfrak{T}_{30},\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{0}(\mathfrak{T}_{30},\langle 4,2,2\rangle)} (au+bv+cw)^4 \\
= \; & a(\mathfrak{T}_{30},\mathcal{L}_{32})\cdot 6480(4u^2+4v^2+4w^2+4uv+4uw+2vw)^2,
\end{aligned}$$

and

$$(4.13)$$

$$\begin{aligned}
& a((\mathfrak{T}_{30},\{1,1,0\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{112}(\mathfrak{T}_{30},\langle 1,1,0\rangle)} (au+bv+cw)^6 \\
& +a((\mathfrak{T}_{30},\{2,1,1\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{96}(\mathfrak{T}_{30},\langle 2,1,1\rangle)} (au+bv+cw)^6 \\
& +a((\mathfrak{T}_{30},\{2,2,1\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{84}(\mathfrak{T}_{30},\langle 2,2,1\rangle)} (au+bv+cw)^6 \\
& +a((\mathfrak{T}_{30},\{2,2,2\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{80}(\mathfrak{T}_{30},\langle 2,2,2\rangle)} (au+bv+cw)^6 \\
& +a((\mathfrak{T}_{30},\{2,2,0\},2),\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{64}(\mathfrak{T}_{30},\langle 2,2,0\rangle)} (au+bv+cw)^6 \\
& +a(\mathfrak{T}_{30},\mathcal{L}_{32}) \sum_{\langle a,b,c\rangle\in C_{0}(\mathfrak{T}_{30},\langle 4,2,2\rangle)} (au+bv+cw)^6 \\
= \; & a(\mathfrak{T}_{30},\mathcal{L}_{32})\cdot 3600(4u^2+4v^2+4w^2+4uv+4uw+2vw)^3
\end{aligned}$$

Equations (4.11),(4.12) and (4.13) are polynomial identities on the variables u, v, w, and from them we have linear equations on the quantities $a((\mathfrak{T}_{30},\{1,1,0\},2),\mathcal{L}_{32}),\cdots,a((\mathfrak{T}_{30},\{2,2,0\},2),\mathcal{L}_{32})$ which enable us to obtain the exact relations among those quantities such as (4.8).

The results of Subsubsection 4.1.3 below can be justified by a similar way to that of the present subsubsection. Therefore we can avoid the way that passes the formulas (4.15) and we go directly to Proposition 4.2 without appealing to (MTA).

4.1.3 Extending $\mathfrak{T}_{31}$

When we use $\mathfrak{T}_{31}$ The following sets:

$$\begin{aligned}
&C_{144}(\mathfrak{T}_{31},\langle 0,0,0\rangle)=\{\langle 0,0,0\rangle\}\\
&C^{(+)}_{132}(\mathfrak{T}_{31},\langle 0,0,1\rangle)=\{\langle 0,0,1\rangle,\langle 0,1,0\rangle\},\\
&C^{(+)}_{129}(\mathfrak{T}_{31},\langle 1,1,1\rangle)=\{\langle 1,1,1,\rangle,\langle 1,1,0,\rangle,\langle 1,0,1,\rangle,\langle 1,0,0,\rangle\},\\
&C^{(+)}_{120}(\mathfrak{T}_{31},\langle 0,1,1\rangle)=\{\langle 0,1,1,\rangle,\langle 0,1,-1\rangle\},\\
&C^{(+)}_{108}(\mathfrak{T}_{31},\langle 2,1,1\rangle)=\{\langle 2,1,1\rangle\},\\
&C^{(+)}_{105}(\mathfrak{T}_{31},\langle 1,2,1\rangle)=\{\langle 1,2,1\rangle,\langle 1,2,0\rangle,\langle 1,1,2\rangle,\langle 1,1,-1\rangle,\\
&\qquad\qquad\langle 1,0,2\rangle,\langle 1,0,-1\rangle,\langle 1,-1,1\rangle,\langle 1,-1,0\rangle\},\\
&C^{(+)}_{96}(\mathfrak{T}_{31},\langle 2,2,1\rangle)=\{\langle 2,2,1\rangle,\langle 2,1,2\rangle,\langle 2,1,0\rangle,\\
&\qquad\qquad\langle 2,0,1\rangle,\langle 0,2,0\rangle,\langle 0,0,2\rangle\},\\
&C^{(+)}_{84}(\mathfrak{T}_{31},\langle 2,2,2\rangle)=\{\langle 2,2,2\rangle,\langle 2,2,0\rangle,\langle 2,0,2\rangle,\langle 2,0,0\rangle,\\
&\qquad\qquad\langle 0,2,1\rangle,\langle 0,2,-1\rangle,\langle 0,1,2\rangle,\langle 0,1,-2\rangle\},\\
&C^{(+)}_{81}(\mathfrak{T}_{31},\langle 1,2,2\rangle)=\{\langle 1,2,2\rangle,\langle 1,2,-1\rangle,\langle 1,-1,2\rangle,\langle 1,-1,-1\rangle\},\\
&C^{(+)}_{0}(\mathfrak{T}_{31},\langle 4,2,2\rangle)=\{\langle 4,2,2\rangle,\langle 2,4,1\rangle,\langle 2,1,4\rangle,\langle 2,1,-2\rangle,\langle 2,-2,1\rangle\},
\end{aligned}$$

exhaust all admissible sets for $\mathfrak{T}_{31}$ We set $\boldsymbol{\alpha}=u\mathbf{x}_1+v\mathbf{x}_2+w\mathbf{x}_3$, where u,v,w are real independent variables and $\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4$ satisfying $[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{31}$. Then it holds that $(\boldsymbol{\alpha},\boldsymbol{\alpha})=4u^2+4v^2+4w^2+4uv+4uw+2vw$. The left-hand side of (2.4) is

(4.14)

$$\begin{aligned}
&\sum_{\mathbf{z}\in\Lambda_4}(\mathbf{z},\boldsymbol{\alpha})^2\\
&=\sum_{\langle a,b,c\rangle\in C_{132}(\mathfrak{T}_{31},\langle 0,0,1\rangle)}\lambda_{a,b,c}(au+bv+cw)^2+\sum_{\langle a,b,c\rangle\in C_{129}(\mathfrak{T}_{31},\langle 1,1,1\rangle)}\lambda_{a,b,c}(au+bv+cw)^2\\
&\quad+\sum_{\langle a,b,c\rangle\in C_{120}(\mathfrak{T}_{31},\langle 0,1,1\rangle)}\lambda_{a,b,c}(au+bv+cw)^2+\sum_{\langle a,b,c\rangle\in C_{108}(\mathfrak{T}_{31},\langle 2,1,1\rangle)}\lambda_{a,b,c}(au+bv+cw)^2\\
&\quad+\sum_{\langle a,b,c\rangle\in C_{105}(\mathfrak{T}_{31},\langle 1,2,1\rangle)}\lambda_{a,b,c}(au+bv+cw)^2+\sum_{\langle a,b,c\rangle\in C_{96}(\mathfrak{T}_{31},\langle 2,2,1\rangle)}\lambda_{a,b,c}(au+bv+cw)^2\\
&\quad+\sum_{\langle a,b,c\rangle\in C_{84}(\mathfrak{T}_{31},\langle 2,2,2\rangle)}\lambda_{a,b,c}(au+bv+cw)^2+\sum_{\langle a,b,c\rangle\in C_{81}(\mathfrak{T}_{31},\langle 1,2,2\rangle)}\lambda_{a,b,c}(au+bv+cw)^2\\
&\quad+\sum_{\langle a,b,c\rangle\in C_{0}(\mathfrak{T}_{31},\langle 4,2,2\rangle)}(au+bv+cw)^2\\
&=2\lambda_{0,0,1}(w^2+v^2)+2\lambda_{1,1,1}((u+v+w)^2+(u+v)^2+(u+w)^2+u^2)\\
&\quad 2\lambda_{0,1,1}((v+w)^2+(v-w)^2)+2\lambda_{2,1,1}(2u+v+w)^2\\
&\quad+2\lambda_{1,2,1}((u+2v+w)^2+(u+2v)^2+(u+v+2w)^2+(u+v-w)^2+(u+2w)^2\\
&\quad+(u-w)^2+(u-v+w)^2+(u-v)^2)+2\lambda_{2,2,1}((2u+2v+w)^2+(2u+v+2w)^2\\
&\quad+(2u+v)^2+(2u+w)^2+(2v)^2+(2w)^2)+2\lambda_{2,2,2}((2u+2v+2w)^2+(2u+2v)^2\\
&\quad+(2u+2w)^2+(2u)^2+(2v+w)^2+(2v-w)^2+(v+2w)^2+(v-2w)^2)\\
&\quad+2\lambda_{1,2,2}((u+2v+2w)^2+(u+2v-w)^2+(u-v+2w)^2+(u-v-w)^2\\
&\quad+2((4u+2v+2w)^2+(2u+4v+w)^2+(2u+v+4w)^2+(2u+v-2w)^2+(2u-2v+w)^2)\\
&=18360(4u^2+4v^2+4w^2+4uv+4uw+2vw).
\end{aligned}$$

By using the equation (2.5) we obtain

$$
\begin{aligned}
& 2\lambda_{0,0,1}(w^4+v^4)+2\lambda_{1,1,1}((u+v+w)^4+(u+v)^4+(u+w)^4+u^4) \\
& 2\lambda_{0,1,1}((v+w)^4+(v-w)^4)+2\lambda_{2,1,1}(2u+v+w)^4 \\
& +2\lambda_{1,2,1}((u+2v+w)^4+(u+2v)^4+(u+v+2w)^4+(u+v-w)^4+(u+2w)^4 \\
& +(u-w)^4+(u-v+w)^4+(u-v)^4)+2\lambda_{2,2,1}((2u+2v+w)^4+(2u+v+2w)^4 \\
& +(2u+v)^4+(2u+w)^4+(2v)^4+(2w)^4)+2\lambda_{2,2,2}((2u+2v+2w)^4+(2u+2v)^4 \\
& +(2u+2w)^4+(2u)^4+(2v+w)^4+(2v-w)^4+(v+2w)^4+(v-2w)^4) \\
& +2\lambda_{1,2,2}((u+2v+2w)^4+(u+2v-w)^4+(u-v+2w)^4+(u-v-w)^4 \\
& +2((4u+2v+2w)^4+(2u+4v+w)^4+(2u+v+4w)^4+(2u+v-2w)^4+(2u-2v+w)^4) \\
= \; & 6480(4u^2+4v^2+4w^2+4uv+4uw+2vw)^2.
\end{aligned}
$$

By using the equation (2.6) we obtain

$$
\begin{aligned}
& 2\lambda_{0,0,1}(w^6+v^6)+2\lambda_{1,1,1}((u+v+w)^6+(u+v)^6+(u+w)^6+u^6) \\
& 2\lambda_{0,1,1}((v+w)^6+(v-w)^6)+2\lambda_{2,1,1}(2u+v+w)^6 \\
& +2\lambda_{1,2,1}((u+2v+w)^6+(u+2v)^6+(u+v+2w)^6+(u+v-w)^6+(u+2w)^6 \\
& +(u-w)^6+(u-v+w)^6+(u-v)^6)+2\lambda_{2,2,1}((2u+2v+w)^6+(2u+v+2w)^6 \\
& +(2u+v)^6+(2u+w)^6+(2v)^6+(2w)^6)+2\lambda_{2,2,2}((2u+2v+2w)^6+(2u+2v)^6 \\
& +(2u+2w)^6+(2u)^6+(2v+w)^6+(2v-w)^6+(v+2w)^6+(v-2w)^6) \\
& +2\lambda_{1,2,2}((u+2v+2w)^6+(u+2v-w)^6+(u-v+2w)^6+(u-v-w)^6 \\
& +2((4u+2v+2w)^6+(2u+4v+w)^6+(2u+v+4w)^6+(2u+v-2w)^6+(2u-2v+w)^6) \\
= \; & 3600(4u^2+4v^2+4w^2+4uv+4uw+2vw)^3.
\end{aligned}
$$

Again we get the following expressions:

$$
(4.15) \qquad \begin{cases}
\lambda_{0,0,1}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) &= 9576+2\lambda_{2,2,2}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3), \\
\lambda_{1,1,1}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) &= 7074-\lambda_{2,2,2}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3), \\
\lambda_{0,1,1}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) &= 2700-2\lambda_{2,2,2}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3), \\
\lambda_{2,1,1}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) &= 552+4\lambda_{2,2,2}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3), \\
\lambda_{1,2,1}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) &= 414+\lambda_{2,2,2}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3), \\
\lambda_{2,2,1}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) &= 171-2\lambda_{2,2,2}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3), \\
\lambda_{2,2,2}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) &= \lambda_{2,2,2}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3), \\
\lambda_{1,2,2}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) &= 34-\lambda_{2,2,2}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3).
\end{cases}
$$

As to $\lambda_{0,0,0}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$ the following fact, which is analogous to the relation directly after (4.6), is useful.

$$\begin{aligned}
&\lambda_{0,0,0} + \sum_{\langle a,b,c\rangle\in C_{132}(\mathfrak{T}_{31},\langle 0,0,1\rangle)} \lambda_{a,b,c} + \sum_{\langle a,b,c\rangle\in C_{129}(\mathfrak{T}_{31},\langle 1,1,1\rangle)} \lambda_{a,b,c}\\
&+ \sum_{\langle a,b,c\rangle\in C_{120}(\mathfrak{T}_{31},\langle 0,1,1\rangle)} \lambda_{a,b,c} + \sum_{\langle a,b,c\rangle\in C_{108}(\mathfrak{T}_{31},\langle 2,1,1\rangle)} \lambda_{a,b,c}\\
&+ \sum_{\langle a,b,c\rangle\in C_{105}(\mathfrak{T}_{31},\langle 1,2,1\rangle)} \lambda_{a,b,c} + \sum_{\langle a,b,c\rangle\in C_{96}(\mathfrak{T}_{31},\langle 2,2,1\rangle)} \lambda_{a,b,c}\\
&+ \sum_{\langle a,b,c\rangle\in C_{84}(\mathfrak{T}_{31},\langle 2,2,2\rangle)} \lambda_{a,b,c} + \sum_{\langle a,b,c\rangle\in C_{81}(\mathfrak{T}_{31},\langle 1,2,2\rangle)} \lambda_{a,b,c}\\
&+ \sum_{\langle a,b,c\rangle\in C_{0}(\mathfrak{T}_{31},\langle 4,2,2\rangle)}\\
= \;& 146880.
\end{aligned}$$

From this we have

$$\lambda_{0,0,0}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = 31122.$$

Proposition 4.2. *For a 32-dimensional even unimodular extremal lattice $\mathcal{L}_{32}$ we have the following Fourier coefficients relations:*

(4.16)

$$\begin{aligned}
a((\mathfrak{T}_{31},\{0,0,0\},2),\mathcal{L}_{32}) &= 31122\cdot 163189555200,\\
a((\mathfrak{T}_{31},\{0,0,1/2\},2),\mathcal{L}_{32}) &= 9576\cdot 163189555200 + 2\cdot a((\mathfrak{T}_{31},\{1,1,1\},2),\mathcal{L}_{32}),\\
a((\mathfrak{T}_{31},\{1/2,1/2,1/2\},2),\mathcal{L}_{32}) &= 7074\cdot 163189555200 - a((\mathfrak{T}_{31},\{1,1,1\},2),\mathcal{L}_{32}),\\
a((\mathfrak{T}_{31},\{0,1/2,1/2\},2),\mathcal{L}_{32}) &= 2700\cdot 163189555200 - 2\cdot a((\mathfrak{T}_{31},\{1,1,1\},2),\mathcal{L}_{32}),\\
a((\mathfrak{T}_{31},\{1,1/2,1/2\},2),\mathcal{L}_{32}) &= 552\cdot 163189555200 + 4\cdot a((\mathfrak{T}_{31},\{1,1,1\},2),\mathcal{L}_{32}),\\
a((\mathfrak{T}_{31},\{1/2,1,1/2\},2),\mathcal{L}_{32}) &= 414\cdot 163189555200 + a((\mathfrak{T}_{31},\{1,1,1\},2),\mathcal{L}_{32}),\\
a((\mathfrak{T}_{31},\{1,1,1/2\},2),\mathcal{L}_{32}) &= 171\cdot 163189555200 - 2\cdot a((\mathfrak{T}_{31},\{1,1,1\},2),\mathcal{L}_{32}),\\
a((\mathfrak{T}_{31},\{1/2,1/2,0\},2),\mathcal{L}_{32}) &= 34\cdot 163189555200 - a((\mathfrak{T}_{31},\{1,1,1\},2),\mathcal{L}_{32}).
\end{aligned}$$

Proof. The proof is very similar to that of Proposition 4.1. This time we have

$$\begin{aligned}
a((\mathfrak{T}_{31},\{0,0,1/2\},2),\mathcal{L}_{32}) &= \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4\\ [\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{31}}} \lambda_{0,0,1}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)\\
&= \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_4\\ [\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2\mathfrak{T}_{31}}} (9576 + 2\cdot\lambda_{2,2,2}(2\mathfrak{T}_{31};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3))\\
&= 9576\cdot 163189555200 + 2\cdot a((\mathfrak{T}_{31},\{1,1,1\},2),\mathcal{L}_{32}).
\end{aligned}$$

Other equations are derived in the same way. □

Remark 4. *The sequence* $18,-10,8,4,-2,1$ *which are the coefficients of* $a((\mathfrak{T}_{30},\{1,1,1\},2),\mathcal{L}_{32})$ *in Proposition* 4.1*, and the sequence* $0,2,-1,-2,4,1,-2,1$ *which are the coefficients of* $a((\mathfrak{T}_{31},\{1,1,1\},2),\mathcal{L}_{32})$ *in Proposition* 4.2 *are the Fourier coefficients of the square of Schottky modular form J (up to constant multiple). (See Table* 6 *in Section* 4.4*). This means that*

Propositions 4.1 *and* 4.2 *are more concrete realizations of Theorem* 4.4 *below which is due to Salvati Manni.*

Table 5. A Dictionary of Extended Quadratic Matrices and its Reduced Forms

D	representative	reduced form
*64	$(\mathfrak{T}_{30}, \{1,1,0\}, 2)$	(2,2,2,2,0,0,0,2,2,2)
*80	$(\mathfrak{T}_{30}, \{1,1,1\}, 2)$	(2,2,2,2,2,0,0,2,0,2)
81	$(\mathfrak{T}_{31}, \{0,0,0\}, 2)$	(2,2,2,2,1,1,1,2,2,-1)
84	$(\mathfrak{T}_{30}, \{1,1,1/2\}, 2)$	(2,2,2,2,1,0,0,2,2,2)
84	$(\mathfrak{T}_{31}, \{1,1,1\}, 2)$	(2,2,2,2,1,0,0,2,2,2)
96	$(\mathfrak{T}_{30}, \{1,1/2,1/2\}, 2)$	(2,2,2,2,2,1,-1,0,0,2)
96	$(\mathfrak{T}_{31}, \{1,1/2,1/2\}, 2)$	(2,2,2,2,2,1,-1,0,0,2)
105	$(\mathfrak{T}_{31}, \{1/2,1,1/2\}, 2)$	(2,2,2,2,2,1,0,0,1,2)
108	$(\mathfrak{T}_{31}, \{0,0,0\}, 2)$	(2,2,2,2,2,1,-1,-1,1,-1)
112	$(\mathfrak{T}_{30}, \{1/2,1/2,0\}, 2)$	(2,2,2,2,2,1,0,2,0,0)
116	$(\mathfrak{T}_{30}, \{1/2,1/2,1/2\}, 2)$	(2,2,2,2,2,1,0,0,2,0)
120	$(\mathfrak{T}_{31}, \{0,1/2,1/2\}, 2)$	(2,2,2,2,1,1,1,2,2,0)
121		(2,2,2,2,2,1,0,1,1,2)
125		(2,2,2,2,1,1,-1,-1,1,1)
*128	$(\mathfrak{T}_{30}, \{0,0,0\}, 2)$	(2,2,2,2,0,0,0,2,2,0)
128		(2,2,2,2,2,1,0,0,0,2)
129	$(\mathfrak{T}_{31}, \{1/2,1/2,1/2\}, 2)$	(2,2,2,2,1,1,1,1,2,2)
132	$(\mathfrak{T}_{31}, \{0,0,1/2\}, 2)$	(2,2,2,2,2,1,-1,0,0,1)

We may merge Propositions 4.1 and 4.2 into the following unified proposition.

Proposition 4.3. *We have*

$$\begin{aligned}
a((\mathfrak{T}_{31}, \{0,0,0\}, 2), \mathcal{L}_{32}) &= 5078785336934400, \\
a((\mathfrak{T}_{30}, \{0,0,0\}, 2), \mathcal{L}_{32}) &= 31992 \cdot 31144435200 + 18 \cdot a((\mathfrak{T}_{30}, \{1,1,1\}, 2), \mathcal{L}_{32}), \\
a((\mathfrak{T}_{31}, \{0,0,1/2\}, 2), \mathcal{L}_{32}) &= 1576655887564800 - 4 \cdot a((\mathfrak{T}_{30}, \{1,1,1\}, 2), \mathcal{L}_{32}), \\
a((\mathfrak{T}_{31}, \{1/2,1/2,1/2\}, 2), \mathcal{L}_{32}) &= 1147426560000000 + 2 \cdot a((\mathfrak{T}_{30}, \{1,1,1\}, 2), \mathcal{L}_{32}), \\
a((\mathfrak{T}_{31}, \{0,1/2,1/2\}, 2), \mathcal{L}_{32}) &= 426659092070400 + 4 \cdot a((\mathfrak{T}_{30}, \{1,1,1\}, 2), \mathcal{L}_{32}), \\
a((\mathfrak{T}_{30}, \{1/2,1/2,1/2\}, 2), \mathcal{L}_{32}) &= 9312 \cdot 31144435200 - 10 \cdot a((\mathfrak{T}_{30}, \{1,1,1\}, 2), \mathcal{L}_{32}), \\
a((\mathfrak{T}_{30}, \{1/2,1/2,0\}, 2), \mathcal{L}_{32}) &= 4992 \cdot 31144435200 + 8 \cdot a((\mathfrak{T}_{30}, \{1,1,1\}, 2), \mathcal{L}_{32}), \\
a((\mathfrak{T}_{31}, \{1,1/2,1/2\}, 2), \mathcal{L}_{32}) &= 117986048409600 - 8 \cdot a((\mathfrak{T}_{30}, \{1,1,1\}, 2), \mathcal{L}_{32}), \\
a((\mathfrak{T}_{31}, \{1/2,1,1/2\}, 2), \mathcal{L}_{32}) &= 74536829337600 - 2 \cdot a((\mathfrak{T}_{30}, \{1,1,1\}, 2), \mathcal{L}_{32}), \\
a((\mathfrak{T}_{30}, \{1,1,1/2\}, 2), \mathcal{L}_{32}) &= 448 \cdot 31144435200 + 4 \cdot a((\mathfrak{T}_{30}, \{1,1,1\}, 2), \mathcal{L}_{32}), \\
a((\mathfrak{T}_{30}, \{1,1,1/2\}, 2), \mathcal{L}_{32}) &= 224 \cdot 31144435200 - 2 \cdot a((\mathfrak{T}_{30}, \{1,1,1\}, 2), \mathcal{L}_{32}), \\
a((\mathfrak{T}_{31}, \{1/2,1,1\}, 2), \mathcal{L}_{32}) &= -1427908608000 + 2 \cdot a((\mathfrak{T}_{30}, \{1,1,1\}, 2), \mathcal{L}_{32}), \\
a((\mathfrak{T}_{30}, \{1,1,0\}, 2), \mathcal{L}_{32}) &= -54 \cdot 31144435200 + a((\mathfrak{T}_{30}, \{1,1,1\}, 2), \mathcal{L}_{32}).
\end{aligned}$$

Proof. We observe that two matrices $(\mathfrak{T}_{30}, \{1,1,1/2\}, 2)$ and $(\mathfrak{T}_{31}, \{1,1,1\}, 2)$ are integrally equivalent matrices with the common discriminant 84. Therefore by the equality (3.2) we have

$$a((\mathfrak{T}_{30}, \{1,1,1/2\}, 2), \mathcal{L}_{32}) = a((\mathfrak{T}_{31}, \{1,1,1\}, 2), \mathcal{L}_{32}).$$

By Proposition 4.1 one has

$$a((\mathfrak{T}_{31},\{1,1,1\},2),\mathcal{L}_{32}) = 224\cdot 31144435200 - 2\cdot a((\mathfrak{T}_{30},\{1,1,1\},2),\mathcal{L}_{32}). \tag{4.17}$$

This equation is a link between Proposition 4.1 and Proposition 4.2. The rest of the proposition follows from this link and Proposition 4.2. □

Remark 5. *Some readers may be aware of the equivalence of two matrices* $(\mathfrak{T}_{30},\{1,1/2,1/2\},2)$ *and* $(\mathfrak{T}_{31},\{1,1,1/2\},2)$ *with the common discriminant* 96*. This equivalence may lead to another equation between* $a((\mathfrak{T}_{30},\{1,1,1\},2),\mathcal{L}_{32})$ *and* $a((\mathfrak{T}_{31},\{1,1,1\},2),\mathcal{L}_{32})$*. However one may verify that this condition overlaps with* (4.17)*.*

4.2 Pan theta series

As to the range of the values of $a(\mathfrak{T}_{40},\mathcal{L}_{32})$ we have an elementary estimate.

Theorem 4.4. *Let* $\mathcal{L}_{32}$ *be any* 32*-dimensional even unimodular extremal lattice, then it holds that*

$$169799500800 = 54\cdot 3144435200 \le a(\mathfrak{T}_{40},\mathcal{L}_{32}) \le 112\cdot 3144435200 = 352176742400.$$

Proof. By the non-negative of the Fourier coefficients $a((\mathfrak{T}_{30},\{1,1,0\},2),\mathcal{L}_{32}), a((\mathfrak{T}_{30},\{1,1,1/2\},2),\mathcal{L}_{32})$ and by Proposition 4.1 we have the inequalities:

$$\begin{aligned} a((\mathfrak{T}_{30},\{1,1,1\},2) &\ge 54\cdot 31144435200, \\ &\text{and} \\ a((\mathfrak{T}_{30},\{1,1,1\},2),\mathcal{L}_{32}) &\le 112\cdot 31144435200. \end{aligned}$$

This implies Theorem. □

Salvati Manni [30] Theorem 3 showed

Theorem 4.5 (Salvati Manni)**.** *Let us assume* $N = 32, 48$ (*the dimensions of the lattices*)*; then about the theta series associated to extremal lattices we can say*

(i) *it is unique in degree* 3*,*
(ii) *in degree* 4 *their difference is, up to a multiplicative constant(possibly* 0*), equal to a power of Schottky's polynomial* J*.*

We normalize that the Fourier coefficient of J^2 at the index $\mathfrak{T}_{40}$ is 1.

Theorem 4.6. *Let* $\Theta_4(Z,\mathcal{L}_{32}) = \sum_T a(T,\mathcal{L}_{32})e^{2\pi i\sigma(TZ)}$ *be the Fourier expansion of Siegel theta series of degree* 4 *for any even unimodular extremal* 32*-dimensional lattice* $\mathcal{L}_{32}$*.*

(1) $\Theta_4(Z,\mathcal{L}_{32})$ *is uniquely determined by the value of the Fourier coefficient at the index* $\mathfrak{T}_{40} = (\mathfrak{T}_{30},\{1,1,1\},2)$*.*

(2) The series defined by

$$P\Theta_4(Z) = \Theta_4(Z,\mathcal{L}_{32}) - a(\mathfrak{T}_{40},\mathcal{L}_{32})J^2$$

is a Siegel modular form of degree 4 *and weight* 16*, and independent of the choice of extremal lattice* $\mathcal{L}_{32}$*.*

Proof. (1) follows from (2). The latter part of (2) can be obtained from Salvati Manni's Theorem. Indeed, that Theorem says the equality

$$\Theta_4(Z, \mathcal{L}_{32}^{(1)}) - \Theta_4(Z, \mathcal{L}_{32}^{(2)}) = cJ^2$$

holds for certain constant c. We compare the both sides of the above equation at the index $\mathfrak{T}_{40}$ in their Fourier expansions. The left hand equals $(a(\mathfrak{T}_{40}, \mathcal{L}_{32}^{(1)}) - a(\mathfrak{T}_{40}, \mathcal{L}_{32}^{(2)}))$ whereas the right hand equals c. This implies that

$$\Theta_4(Z, \mathcal{L}_{32}^{(1)}) - \Theta_4(Z, \mathcal{L}_{32}^{(2)}) = (a(\mathfrak{T}_{40}, \mathcal{L}_{32}^{(1)}) - a(\mathfrak{T}_{40}, \mathcal{L}_{32}^{(2)}))J^2,$$

or

$$\Theta_4(Z, \mathcal{L}_{32}^{(1)}) - a(\mathfrak{T}_{40}, \mathcal{L}_{32}^{(1)})J^2 = \Theta_4(Z, \mathcal{L}_{32}^{(2)}) - a(\mathfrak{T}_{40}, \mathcal{L}_{32}^{(2)})J^2.$$

for any two 32-dimensional even unimodular extremal lattices $\mathcal{L}_{32}^{(1)}, \mathcal{L}_{32}^{(2)}$. □

We call $P\Theta_4(Z)$ a pan theta series.

Remark 6. *At present we have only one method to compute the Fourier coefficients of pan theta series $P\Theta_4(Z)$, namely the method to compute them step by step using Hecke-Schöneberg equations along the line in obtaining Propositions* 4.1 *and* 4.2*. If we could find a means to approach this series from a different way, it may be nice.*

Here we give a table of the Fourier coefficients of pan theta series.

Table 6. Table of the Fourier coefficients of pan theta series

D	reduced form	$P\Theta_4(Z)$
*64	(2,2,2,2,0,0,0,2,2,2)	-1681799500800
*80	(2,2,2,2,2,0,0,2,0,2)	0
81	(2,2,2,2,1,1,1,2,2,-1)	-1427908608000
84	(2,2,2,2,1,0,0,2,2,2)	6976353484800
96	(2,2,2,2,2,1,-1,0,0,2)	13952706969600
105	(2,2,2,2,2,1,0,0,1,2)	74536829337600
108	(2,2,2,2,2,1,-1,-1,1,-1)	117986048409600
112	(2,2,2,2,2,1,0,2,0,0)	155473020518400
116	(2,2,2,2,2,1,0,0,2,0)	290016980582400
120	(2,2,2,2,1,1,1,2,2,0)	426659092070400
121	(2,2,2,2,2,1,0,1,1,2)	449738757734400
125	(2,2,2,2,1,1,-1,-1,1,1)	737143539793920
*128	(2,2,2,2,0,0,0,2,2,0)	996372770918400
128	(2,2,2,2,2,1,0,0,0,2)	1064392217395200
129	(2,2,2,2,1,1,1,1,2,2)	1147426560000000
132	(2,2,2,2,2,1,-1,0,0,1)	1576655887564800

4.3 A Very Small Table of the Fourier Coefficients of the Square of Schottky Modular Form

The Schottky modular form J is a unique Siegel cusp form of degree 4 and weight 8. At present we have some methods to compute the Fourier coefficients of J. For instance we pick up two Siegel series of degree 4 associated with root lattices of type $E_8 \oplus E_8$ and

D_{16}. Then the difference of these two series is a constant multiple of the Schottky modular form J (c.f. [12], [13]). Beside this we have a method to get Siegel theta series of degree 4 from the quadriweight enumerator of doubly even self-dual codes of types $e_8 \oplus e_8$ and d_{16} (c.f. [5]). Kawamura [14] has computed the Fourier coefficients of many Siegel cusp forms of various weights, including Schottky modular form, using the Ikeda lift. From the Fourier coefficients of J we can compute the Fourier coefficients of the square J^2. We give these as the following table.

Table 7.
Fourier coefficients of the square of Schottky modular form J.

D	reduced form	J^2
*64	(2,2,2,2,0,0,0,2,2,2)	1
*80	(2,2,2,2,2,0,0,2,0,2)	1
81	(2,2,2,2,1,1,1,2,2,-1)	2
84	(2,2,2,2,1,0,0,2,2,2)	-2
96	(2,2,2,2,2,1,-1,0,0,2)	4
105	(2,2,2,2,2,1,0,0,1,2)	-2
108	(2,2,2,2,2,1,-1,-1,1,-1)	-8
112	(2,2,2,2,2,1,0,2,0,0)	8
116	(2,2,2,2,2,1,0,0,2,0)	-10
120	(2,2,2,2,1,1,1,2,2,0)	4
121	(2,2,2,2,2,1,0,1,1,2)	20
125	(2,2,2,2,1,1,-1,-1,1,1)	12
*128	(2,2,2,2,0,0,0,2,2,0)	18
128	(2,2,2,2,2,1,0,0,0,2)	-16
129	(2,2,2,2,1,1,1,1,2,2)	2
132	(2,2,2,2,2,1,-1,0,0,1)	-4
140	(2,2,2,2,1,1,-1,0,0,2)	8
*144	(2,2,2,2,2,0,0,0,0,2)	66
144	(2,2,2,2,2,1,-1,0,0,0)	0
145	(2,2,2,2,2,1,0,-1,-1,1)	-10
153	(2,2,2,2,1,1,0,1,1,2)	-54
156	(2,2,2,2,1,1,1,2,0,0)	48
160	(2,2,2,2,1,1,-1,1,-1,0)	-16

D	reduced form	J^2
160	(2,2,2,2,1,1,0,2,0,0)	-44
161	(2,2,2,2,2,1,0,0,1,0)	46
164	(2,2,2,2,1,0,0,1,0,2)	52
165	(2,2,2,2,1,1,-1,-1,-1,1)	-4
176	(2,2,2,2,1,0,0,0,2,0)	-72
180	(2,2,2,2,1,1,1,-1,1,0)	0
180	(2,2,2,2,1,0,0,0,0,2)	-228
185	(2,2,2,2,1,1,0,1,0,-1)	28
189	(2,2,2,2,1,1,1,1,1,1)	-244
*192	(2,2,2,2,0,0,0,2,0,0)	324
192	(2,2,2,2,1,1,1,1,1,0)	208
192	(2,2,2,2,1,1,0,0,1,1)	104
196	(2,2,2,2,1,1,0,0,1,-1)	264
200	(2,2,2,2,1,1,-1,0,0,0)	-72
201	(2,2,2,2,1,1,1,0,1,0)	-156
208	(2,2,2,2,1,1,0,1,0,0)	80
209	(2,2,2,2,1,1,0,0,1,0)	-304
216	(2,2,2,2,1,1,1,0,0,0)	216
224	(2,2,2,2,1,1,0,0,0,0)	504
225	(2,2,2,2,1,0,0,0,0,1)	1092
240	(2,2,2,2,1,0,0,0,0,0)	-1728
*256	(2,2,2,2,0,0,0,0,0,0)	4104

The reduced quadratic matrices of size 4 which is expressed in the form (4.1) are displayed as $t_{11}, t_{22}, t_{33}, t_{44}, t_{12}, t_{13}, t_{23}, a = t_{14}, b = t_{24}, c = t_{34}$.

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